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The Outer-Automorphic WZW Orbifolds on $\mathfrak{so}(2n)$, including Five Triality Orbifolds on $\mathfrak{so}(8)$

O. Ganor[a], M. B. Halpern[a], C. Helfgott[a], N. A. Obers[b]

[a] *Department of Physics, University of California, Berkeley, California 94720, USA*
and
Theoretical Physics Group, Lawrence Berkeley National Laboratory
University of California, Berkeley, California 94720, USA

[b] *The Niels Bohr Institute, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark*

origa@socrates.berkeley.edu, halpern@physics.berkeley.edu,
helfgott@socrates.berkeley.edu, obers@nbi.dk

Abstract

Following recent advances in the local theory of current-algebraic orbifolds we present the basic dynamics - including the *twisted KZ equations* - of each twisted sector of all outer-automorphic WZW orbifolds on $\mathfrak{so}(2n)$. Physics-friendly Cartesian bases are used throughout, and we are able in particular to assemble two $\mathbb{Z}_3$ triality orbifolds and three S_3 triality orbifolds on $\mathfrak{so}(8)$.

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1 Introduction

In the last few years there has been a quiet revolution in the local theory of *current-algebraic orbifolds*. Building on the discovery of *orbifold affine algebras* [1, 2] in the cyclic permutation orbifolds, Refs. [3, 4, 5] gave the twisted currents and stress tensor in each twisted sector of any current-algebraic orbifold $A(H)/H$ - where $A(H)$ is any current-algebraic conformal field theory [6-11] with a finite symmetry group H. The construction treats all current-algebraic orbifolds at the same time, using the method of *eigenfields* and the *principle of local isomorphisms* to map OPEs in the symmetric theory to OPEs in the orbifold. The orbifold results are expressed in terms of a set of twisted tensors or *duality transformations*, which are discrete Fourier transforms constructed from the eigendata of the *H-eigenvalue problem*.

More recently, the special case of the WZW orbifolds $A_g(H)/H$, $H \subset Aut(g)$ was worked out in further detail [12, 13, 14], introducing the *extended H-eigenvalue problem* and the *linkage relation* to include the *twisted affine primary fields* and the *twisted KZ equations* of the WZW orbifolds. The general form of the twisted KZ equations is reviewed in Sec. 2.

In addition to the operator formulation, the *general WZW orbifold action* was also given in Ref. [12], with applications to special cases in Refs. [12, 13]. The general WZW orbifold action provides the classical description of each sector of every WZW orbifold in terms of appropriate *group orbifold elements*, which are the classical limit of the twisted affine primary fields. Moreover, Ref. [15] gauged the general WZW orbifold action by general twisted gauge groups to obtain the *general coset orbifold action*.

In this paper we illustrate the general formulation of WZW orbifolds by working out another large class of examples in further detail. In particular, we use physics-friendly Cartesian bases to describe the basic dynamics, including the twisted affine Lie algebras, the twisted affine-Sugawara constructions and the twisted KZ equations of all the outer-automorphic WZW orbifolds on $\mathfrak{so}(2n) \cong \mathfrak{spin}(2n)$. This includes two $\mathbb{Z}_3$ triality orbifolds and, somewhat surprisingly, three S_3 triality orbifolds on $\mathfrak{so}(8) \cong \mathfrak{spin}(8)$.

Some remarks about general WZW orbifolds are also included. In particular, Subsec. 4.2 completes the solution of the rectification problem [12] for all the basic twisted right-mover current algebras, and Subsec. 4.3 completes the computation of the scalar twist-field conformal weights for all sectors of all the basic WZW orbifolds.

2 The Twisted KZ Systems of the WZW Orbifolds

We begin by reminding the reader that twisted KZ systems are now known [12, 13, 14] for the correlators in the *scalar twist-field state*[‡1] $|0\rangle_\sigma = \tau_\sigma(0)|0\rangle$

$$\hat{A}_+(\mathcal{T}, z, \sigma) \equiv {}_\sigma\langle 0|\hat{g}_+(\mathcal{T}^{(1)}, z_1, \sigma)\hat{g}_+(\mathcal{T}^{(2)}, z_2, \sigma) \cdots \hat{g}_+(\mathcal{T}^{(N)}, z_N, \sigma)|0\rangle_\sigma \tag{2.1a}$$

$$\hat{A}_-(\mathcal{T}, \bar{z}, \sigma) \equiv {}_\sigma\langle 0|\hat{g}_-(\mathcal{T}^{(1)}, \bar{z}_1, \sigma)\hat{g}_-(\mathcal{T}^{(2)}, \bar{z}_2, \sigma) \cdots \hat{g}_-(\mathcal{T}^{(N)}, \bar{z}_N, \sigma)|0\rangle_\sigma \tag{2.1b}$$

$$\hat{J}_{n(r)\mu}(m + \tfrac{n(r)}{\rho(\sigma)} \geq 0)|0\rangle_\sigma = {}_\sigma\langle 0|\hat{J}_{n(r)\mu}(m + \tfrac{n(r)}{\rho(\sigma)} \leq 0) = 0 \tag{2.1c}$$

[‡1]For the inner-automorphic WZW orbifolds a different set of twisted KZ equations [12] was given for the correlators in the untwisted affine vacuum state.

$$\hat{\bar{J}}_{n(r)\mu}(m+\tfrac{n(r)}{\rho(\sigma)}\le 0)|0\rangle_\sigma = {}_\sigma\langle 0|\hat{\bar{J}}_{n(r)\mu}(m+\tfrac{n(r)}{\rho(\sigma)}\ge 0)=0 \tag{2.1d}$$

$$\sigma=0,\dots,N_c-1 \tag{2.1e}$$

in each sector σ of any WZW orbifold $A_g(H)/H$. Here N_c is the number of conjugacy classes of the original symmetry group H, $\hat{J}_{n(r)\mu}(m+\frac{n(r)}{\rho(\sigma)})$ are the modes of the twisted current algebra [5, 12] of that sector and

$$\hat{g}(\mathcal{T},\bar{z},z,\sigma)=\hat{g}_-(\mathcal{T},\bar{z},\sigma)\hat{g}_+(\mathcal{T},z,\sigma) \tag{2.2}$$

is the *twisted affine primary field* in twisted representation $\mathcal{T}=\mathcal{T}(T,\sigma)$ of sector σ. The sector $\sigma=0$ is the symmetric theory $A_g(H)$, $H\subset \mathrm{Aut}\,g$ where the currents and affine primary fields are untwisted and the scalar twist-field state $|0\rangle_0=|0\rangle$ is the untwisted affine vacuum.

The scalar twist-field states exist for all sectors of all current-algebraic orbifolds. The reason [14] is easy to understand in the equivalent form

$$\hat{J}_{n(r)\mu}(m+\tfrac{n(r)}{\rho(\sigma)}>0)|0\rangle_\sigma=0,\qquad \hat{J}_{0\mu}(0)|0\rangle_\sigma=0\ . \tag{2.3}$$

The first condition holds for any primary state of any (infinite-dimensional) Lie algebra and the second condition restricts our attention to the "s-wave" or trivial representation of the residual (untwisted) symmetry of the sector. For the WZW permutation orbifolds and the outer-automorphic WZW orbifolds on simple g, the scalar twist-field state is also known to be the ground state of each sector.

For each sector σ of any WZW orbifold $A_g(H)/H$, the twisted left-and right-mover KZ systems are

$$\partial_\kappa \hat{A}_+(\mathcal{T},z,\sigma)=\hat{A}_+(\mathcal{T},z,\sigma)\hat{W}_\kappa(\mathcal{T},z,\sigma),\qquad ,\qquad \kappa=1\dots N,\qquad \sigma=0,\dots,N_c-1 \tag{2.4a}$$

$$\hat{W}_\kappa(\mathcal{T},z,\sigma)=2\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)\mu;-n(r),\nu}(\sigma)\left[\sum_{\rho\ne\kappa}\left(\frac{z_\rho}{z_\kappa}\right)^{\frac{\bar{n}(r)}{\rho(\sigma)}}\frac{1}{z_{\kappa\rho}}\mathcal{T}^{(\rho)}_{n(r)\mu}\mathcal{T}^{(\kappa)}_{-n(r),\nu}-\tfrac{\bar{n}(r)}{\rho(\sigma)}\frac{1}{z_\kappa}\mathcal{T}^{(\kappa)}_{n(r)\mu}\mathcal{T}^{(\kappa)}_{-n(r),\nu}\right] \tag{2.4b}$$

$$\bar{\partial}_\kappa \hat{A}_-(\mathcal{T},\bar{z},\sigma)=\hat{\bar{W}}_\kappa(\mathcal{T},\bar{z},\sigma)\hat{A}_-(\mathcal{T},\bar{z},\sigma),\qquad ,\qquad \kappa=1\dots N,\qquad \sigma=0,\dots,N_c-1 \tag{2.5a}$$

$$\hat{\bar{W}}_\kappa(\mathcal{T},\bar{z},\sigma)=2\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)\mu;-n(r),\nu}(\sigma)\left[\sum_{\rho\ne\kappa}\left(\frac{\bar{z}_\rho}{\bar{z}_\kappa}\right)^{\frac{\bar{n}(r)}{\rho(\sigma)}}\frac{1}{\bar{z}_{\kappa\rho}}\mathcal{T}^{(\kappa)}_{n(r)\mu}\mathcal{T}^{(\rho)}_{-n(r),\nu}-\tfrac{\bar{n}(r)}{\rho(\sigma)}\frac{1}{\bar{z}_\kappa}\mathcal{T}^{(\kappa)}_{n(r)\mu}\mathcal{T}^{(\kappa)}_{-n(r),\nu}\right] \tag{2.5b}$$

$$\hat{A}_{+}(\mathcal{T},z,\sigma)\left(\sum_{\rho=1}^{N}\mathcal{T}_{0,\mu}^{(\rho)}\right)=\left(\sum_{\rho=1}^{N}\mathcal{T}_{0,\mu}^{(\rho)}\right)\hat{A}_{-}(\mathcal{T},\bar{z},\sigma)=0,\qquad\forall\ \mu \tag{2.6a}$$

$$\mathcal{T}^{(\rho)}\mathcal{T}^{(\kappa)}\equiv\mathcal{T}^{(\rho)}\otimes\mathcal{T}^{(\kappa)},\qquad z_{\kappa\rho}\equiv z_\kappa-z_\rho,\qquad \bar{z}_{\kappa\rho}\equiv\bar{z}_\kappa-\bar{z}_\rho\ . \tag{2.6b}$$

Here $\rho(\sigma)$ is the order of the automorphism $h_\sigma \in H$ and the integers $n(r)$ and their pullbacks $\bar{n}(r)$ are obtained from the H*-eigenvalue problem* of sector σ. The twisted tensors $\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}(\sigma)$ and $\mathcal{T}=\mathcal{T}(T,\sigma)$ are respectively the *twisted inverse inertia tensor* and the *twisted representation matrices*, formulas for which are given in Ref. [12], and the relations in (2.6a) are the Ward identities of the residual symmetry algebra of each sector σ. In the untwisted sector $\sigma=0$, the twisted KZ systems reduce to the ordinary KZ system [16] of the symmetric theory $A_g(H)$.

The explicit form of the general twisted KZ system (2.4)-(2.6) has so far been worked out for the following cases:

- the WZW permutation orbifolds [12, 13, 14]
- the inner-automorphic WZW orbifolds [14] on simple g
- the (outer-automorphic) charge conjugation orbifold on $\mathfrak{su}(n\geq 3)$ [13].

Ref. [13] also solved the twisted vertex operator equations and the twisted KZ systems in an abelian limit to obtain the twisted vertex operators for each sector of a very large class of abelian orbifolds[‡2]. The WZW permutation orbifolds also exhibit an extended operator algebra including *twisted Virasoro generators* [1, 18, 14] and finally - with emphasis on the WZW permutation orbifolds - the general twisted KZ system has been used to study the reducibility [14] of the general twisted affine primary field.

In what follows, we apply the general theory of WZW orbifolds to work out the basic dynamics of all the outer-automorphic WZW orbifolds on $\mathfrak{so}(2n)$, including the triality orbifolds on $\mathfrak{so}(8)$. The explicit forms of all the relevant twisted KZ systems are found in Subsec. 4.4. Except for our discussion of the rectification problem in Subsec. 4.2 and the action formulation in Subsec. 4.6, we concentrate for brevity on the twisted left-mover systems - but the twisted right-mover systems [12, 13, 14] can be easily evaluated with the same data.

3 The Currents and Symmetries of $\mathfrak{so}(2n)$ and $\mathfrak{so}(8)$

For accessibility in physics we begin in the standard Cartesian basis of $\mathfrak{so}(2n)$, where the OPEs of the currents of affine $\mathfrak{so}(2n)$ are

$$J_{ij}(z)J_{kl}(w)=\frac{2k(e_{ij})_{kl}}{(z-w)^2}+\frac{i}{z-w}(\delta_{jk}J_{il}(w)+\delta_{il}J_{jk}(w)-\delta_{ik}J_{jl}(w)-\delta_{jl}J_{ik}(w))+\mathcal{O}(z-w)^0 \tag{3.1a}$$

[‡2] An abelian twisted KZ equation for the inversion orbifold $x\to -x$ was given earlier in Ref. [17].

$$(e_{ij})_{kl} \equiv \frac{1}{2}(\delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il}), \qquad i,j = 1\dots n, \qquad J_{ij}(z) = -J_{ji}(z) \ . \tag{3.1b}$$

Here we have chosen root length $\psi^2 = 2$ for $\mathfrak{so}(2n \geq 4)$ and the invariant level of $\mathfrak{so}(2n \geq 4)$ is $x = 2k/\psi^2 = k$. The antisymmetric tensors $\{e_{ij}\}$ satisfy

$$(e_{ij})_{kl} J_{kl}(z) = J_{ij}(z), \qquad (e_{ij})_{ij} = \frac{1}{2}, \qquad \mathrm{Tr}(e_{ij}e_{kl}) = -(e_{ij})_{kl} \tag{3.2}$$

and the $n(2n-1)$ independent currents of $\mathfrak{so}(2n)$ are obtained by choosing $1 \leq i < j \leq 2n$. In what follows we discuss the action of the various outer automorphisms (see the table in App. A) of $g = \mathfrak{so}(2n)$ and $\mathfrak{so}(8)$ in the notation of Refs. [3, 5]

$$J_{ij}(z)' = \sum_{k,l} \omega_{ij,kl} J_{kl}(z) \equiv \omega_{ij,kl} J_{kl}(z), \qquad \omega_{ij,kl} = -\omega_{ji,kl} = -\omega_{ij,lk} \tag{3.3}$$

where the matrix $\omega \equiv \omega(h_\sigma)$ is the action of the automorphism $h_\sigma \in H \subset \mathrm{Aut}\, g$ in the adjoint. Moreover, because it will facilitate transition to the orbifold, we will insist in this paper on finding bases in which the action ω of each automorphism is *diagonal*.

3.1 The parity automorphism $\mathbb{P}$

As defined below, the parity automorphism $\mathbb{P}$ on $\mathfrak{so}(2n)$ is a $\mathbb{Z}_2$-type outer automorphism for all $2n \geq 6$. For $2n \geq 8$, $\mathbb{P}$ is equivalent to the exchange of the two rightmost nodes of the D_n Dynkin diagrams and for $\mathfrak{so}(6)$ it is equivalent to the exchange of the outer nodes of the Dynkin diagram of $\mathfrak{su}(4)$. Then we know that, up to a unitary transformation, $\mathbb{P}$ maps $S \leftrightarrow C$ where S and C

$$\mu_S = \frac{1}{2}\sum_{i=1}^{n} \sigma_i e_i, \qquad \sigma_i = \pm, \qquad \text{(even number of +)} \tag{3.4a}$$

$$\mu_C = \frac{1}{2}\sum_{i=1}^{n} \sigma_i e_i, \qquad \sigma_i = \pm, \qquad \text{(odd number of +)} \tag{3.4b}$$

are the Weyl spinor reps of $\mathfrak{spin}(2n)$ with opposite chirality (see also App. B).

To obtain a diagonal basis for $\mathbb{P}$, we label the currents as follows

$$J_{ij}(z) = \{J_{\mu\nu}(z), J_\mu(z) \equiv J_{\mu,2n}(z)\}, \qquad \mu,\nu = 1,\dots,2n-1 \tag{3.5}$$

where the set $\{J_{\mu\nu}(z)\}$ generates an affine $\mathfrak{so}(2n-1)$ at the same invariant level x. In this basis, the OPEs of affine $\mathfrak{so}(2n)$ read

$$J_{\mu\nu}(z)J_{\rho\sigma}(w) = \frac{2k(e_{\mu\nu})_{\rho\sigma}}{(z-w)^2} + \frac{i}{z-w}(\delta_{\nu\rho}J_{\mu\sigma}(w) + \delta_{\mu\sigma}J_{\nu\rho}(w) - \delta_{\mu\rho}J_{\nu\sigma}(w) - \delta_{\nu\sigma}J_{\mu\rho}(w)) + \mathcal{O}(z-w)^0 \tag{3.6a}$$

$$J_{\mu\nu}(z)J_\rho(w) = \frac{i}{z-w}(\delta_{\nu\rho}J_\mu(w) - \delta_{\mu\rho}J_\nu(w)) + \mathcal{O}(z-w)^0 \tag{3.6b}$$

$$J_\mu(z)J_\nu(w) = \frac{k\delta_{\mu\nu}}{(z-w)^2} - \frac{i}{z-w}J_{\mu\nu}(w) + \mathcal{O}(z-w)^0 \ . \tag{3.6c}$$

The automorphism $\mathbb{P}$ acts on the currents as

$$\omega_{\mu\nu,\rho\sigma} = (e_{\mu\nu})_{\rho\sigma}, \qquad \omega_{\mu,\nu} = -(e_{\mu,2n})_{\nu,2n} = -\delta_\mu^\nu \tag{3.7a}$$

$$J_{\mu\nu}(z)' = J_{\mu\nu}(z), \qquad J_\mu(z)' = -J_\mu(z) \tag{3.7b}$$

where the sign reversal of the vector current is a "parity" transformation. This action corresponds to the coset decomposition $g = h + g/h$, where h is the invariant subalgebra and

$$\frac{g}{h} = \frac{\mathfrak{so}(2n)_x}{\mathfrak{so}(2n-1)_x}, \qquad n \geq 3 \tag{3.8}$$

is a symmetric space. As seen in the table of App. A, these symmetric spaces correspond to the outer automorphisms in entry iv) and the "alternate" charge conjugation automorphism (see Ref. [13]) on $\mathfrak{su}(4)$

$$\frac{\mathfrak{so}(6)}{\mathfrak{so}(5)} = \frac{\mathfrak{su}(4)}{C_2} = \frac{\mathfrak{su}(4)}{\mathfrak{sp}(2)} \tag{3.9}$$

given in entry iii) of the table. Note that all $\mathbb{Z}_2$-type automorphisms, inner or outer, correspond to symmetric space decompositions, and vice-versa, but not all symmetric space decompositions correspond to outer automorphisms.

3.2 The charge-conjugation automorphism $\mathbb{C}$

For any Lie g, the charge-conjugation automorphism $\mathbb{C} \in \mathbb{Z}_2$ is equivalent to sign reversal $\{\mu(T) \to -\mu(T)\}$ of all weights μ of any irrep T. Correspondingly, the action T' of $\mathbb{C}$ on any matrix irrep T of Lie g is unitarily equivalent to $\bar{T}$

$$T' \equiv \omega T \cong \bar{T} \equiv -T^{\mathrm{t}} \tag{3.10}$$

where t is matrix transpose. See Ref. [13] for the diagonal action of $\mathbb{C}$ in the standard Cartesian basis of $\mathfrak{su}(n)$.

To obtain a diagonal basis for $\mathbb{C}$ on $\mathfrak{so}(2n)_x$, we label the currents as follows

$$J_{ij}(z) = \{J_{AB}(z), J_{IJ}(z), J_{AI}(z)\} \tag{3.11a}$$

$$i = (A, I), \qquad A = 1 \ldots n, \qquad I = n+1, \ldots, 2n \tag{3.11b}$$

so that the zero modes of the currents $\{J_{AB}(z), J_{IJ}(z)\}$ generate the subalgebra $(\mathfrak{so}(n)\oplus\mathfrak{so}(n)) \subset \mathfrak{so}(2n)$. In this basis, the OPEs of affine $\mathfrak{so}(2n)$ read

$$J_{AB}(z)J_{CD}(w) = \frac{2k(e_{AB})_{CD}}{(z-w)^2} + \frac{i}{z-w}(\delta_{BC}J_{AD}(w) + \delta_{AD}J_{BC}(w) - \delta_{AC}J_{BD}(w) - \delta_{BD}J_{AC}(w)) + \mathcal{O}(z-w)^0 \tag{3.12a}$$

$$J_{IJ}(z)J_{KL}(w) = \frac{2k(e_{IJ})_{KL}}{(z-w)^2} + \frac{i}{z-w}(\delta_{JK}J_{IL}(w) + \delta_{IL}J_{JK}(w) - \delta_{IK}J_{JL}(w) - \delta_{JL}J_{IK}(w)) + \mathcal{O}(z-w)^0 \tag{3.12b}$$

$$J_{AB}(z)J_{IJ}(w) = \mathcal{O}(z-w)^0 \tag{3.12c}$$

$$J_{AB}(z)J_{CI}(w) = \frac{i}{z-w}(\delta_{BC}J_{AI}(w) - \delta_{AC}J_{BI}(w)) + \mathcal{O}(z-w)^0 \tag{3.12d}$$

$$J_{IJ}(z)J_{AK}(w) = \frac{i}{z-w}(\delta_{JK}J_{AI}(w) - \delta_{IK}J_{AJ}(w)) + \mathcal{O}(z-w)^0 \tag{3.12e}$$

$$J_{AI}(z)J_{BJ}(w) = \frac{k\delta_{AB}\delta_{IJ}}{(z-w)^2} - \frac{i}{z-w}(\delta_{AB}J_{IJ}(w) + \delta_{IJ}J_{AB}(w)) + \mathcal{O}(z-w)^0\ . \tag{3.12f}$$

The diagonal action of $\mathbb{C}$ in this basis

$$\omega_{AB,CD} = (e_{AB})_{CD}, \qquad \omega_{IJ,KL} = (e_{IJ})_{KL}, \qquad \omega_{AI,BJ} = -(e_{AI})_{BJ} = -\delta_{AB}\delta_{IJ} \tag{3.13a}$$

$$J_{AB}(z)' = J_{AB}(z), \qquad J_{IJ}(z)' = J_{IJ}(z), \qquad J_{AI}(z)' = -J_{AI}(z) \tag{3.13b}$$

is an automorphism of the OPEs (3.12). The action (3.13) defines the symmetric space

$$\frac{g}{h} = \frac{\mathfrak{so}(2n)_x}{\mathfrak{so}(n)_{x\tau(n)} \oplus \mathfrak{so}(n)_{x\tau(n)}}, \qquad \tau(n) = \begin{cases} 2 & \text{for } n=3 \\ 1 & \text{for } n\geq 4 \end{cases} \tag{3.14}$$

with invariant subalgebra $h = \mathfrak{so}(n)\oplus\mathfrak{so}(n)$.

To see that $\mathbb{C}$ is equivalent to charge conjugation on $\mathfrak{so}(2n) \cong \mathfrak{spin}(2n)$, note that the choice of Cartan subalgebra $\{J_{i,2n-i},\ i = 1\ldots n\}$ is entirely in g/h. Then the action of $\mathbb{C}$ on these Cartan currents is

$$J_{i,2n-i}(z)' = -J_{i,2n-i}(z), \qquad i = 1\ldots n \tag{3.15}$$

and, as required by charge conjugation, $\mathbb{C}$ reverses the sign of all weights as measured by this choice of Cartan subalgebra.

Of course, this choice is equivalent to any other choice of Cartan subalgebra, for example the standard choice $\{J_{2i-1,2i},\ i = 1\ldots n\}$. For $\mathfrak{so}(4r)$ the action of $\mathbb{C}$ on these Cartan currents is

$$J_{2i-1,2i}(z)' = J_{2i-1,2i}(z), \qquad i = 1\ldots n, \qquad n = 2r \tag{3.16}$$

so that sign reversal of all weights is equivalent to no sign reversal and

$$T' \cong \bar{T} \cong T, \qquad \forall\ T \text{ on } \mathfrak{so}(4r) \cong \mathfrak{spin}(4r)\ . \tag{3.17}$$

This is in agreement with the table in App. A, which tells us that

- $\mathbb{C} =$ inner automorphism of $\mathfrak{so}(4r)$ (3.18a)
- $\mathbb{C} \simeq \mathbb{P} =$ outer automorphism of $\mathfrak{so}(4r+2)$ (3.18b)

where $\mathbb{P}$ is the Dynkin automorphism of $\mathfrak{so}(2n)$. More precisely, (3.18b) means that $\omega(\mathbb{C}) = K\omega(\mathbb{P})$ where K is an inner automorphism which acts non-trivially. Another way to understand (3.18) is as follows: Since $\mathbb{C}$ reverses the sign of each weight, it is easy to check from the weights (3.6) that $\mathbb{C}$ maps

$$S \leftrightarrow C \qquad \text{for } \mathfrak{so}(4r+2) \cong \mathfrak{spin}(4r+2) \tag{3.19a}$$

$$S \to S, \qquad C \to C \qquad \text{for } \mathfrak{so}(4r) \cong \mathfrak{spin}(4r) \tag{3.19b}$$

which confirms the statement in (3.18).

3.3 The interpolating automorphisms $\{\mathbb{A}(2n;r)\}$

We consider next a set of $\mathbb{Z}_2$-type automorphisms of $\mathfrak{so}(2n)$

$$\{\mathbb{A}(2n;r)\}, \qquad r = n, \ldots, 2n-1\ : \qquad \frac{g}{h} = \frac{\mathfrak{so}(2n)_x}{\mathfrak{so}(r)_{x\tau(r)} \oplus \mathfrak{so}(2n-r)_{x\tau(2n-r)}} \tag{3.20}$$

which, as we shall see, interpolates between the automorphisms $\mathbb{P}$ and $\mathbb{C}$ described above. More precisely, we find that the "endpoints" of the sequence are

$$\mathbb{A}(2n;n) = \mathbb{C}, \qquad \mathbb{A}(2n;2n-1) = \mathbb{P} \tag{3.21}$$

as one might expect by comparison of the symmetric spaces in (3.20) with those in (3.8) and (3.14).

To describe the action of $\{\mathbb{A}(2n;r)\}$, it is convenient to employ the same notation used above for $\mathbb{C}$, now with the more general ranges

$$\{\mathbb{A}(2n;r)\}, \qquad r = n, \ldots, 2n-1 \tag{3.22a}$$

$$J_{ij}(z) = \{J_{AB}(z), J_{IJ}(z), J_{AI}(z)\}, \tag{3.22b}$$

$$i = (A, I), \qquad A = 1 \dots r, \qquad I = r+1, \dots, 2n \tag{3.22c}$$

$$\omega_{AB,CD} = (e_{AB})_{CD}, \qquad \omega_{IJ,KL} = (e_{IJ})_{KL}, \qquad \omega_{AI,BJ} = -(e_{AI})_{BJ} = -\delta_{AB}\delta_{IJ} \tag{3.22d}$$

$$J_{AB}(z)' = J_{AB}(z), \qquad J_{IJ}(z)' = J_{IJ}(z), \qquad J_{AI}(z)' = -J_{AI}(z)\ . \tag{3.22e}$$

Bearing these ranges in mind, the OPEs (3.12) are in a diagonal basis for all $\{\mathbb{A}(2n;r)\}$.

According to the table of App. A, the automorphism $\mathbb{A}(2n;r)$ is an outer automorphism of $\mathfrak{so}(2n)$ only when r is odd, and an inner automorphism when r is even. This gives for example the $\mathbb{Z}_2$-type outer automorphisms

$$\mathbb{A}(6;3) \quad : \quad \frac{\mathfrak{so}(6)_x}{\mathfrak{so}(3)_{2x} \oplus \mathfrak{so}(3)_{2x}}, \qquad \mathbb{P} = \mathbb{A}(6;5) \quad : \quad \frac{\mathfrak{so}(6)_x}{\mathfrak{so}(5)_x} \tag{3.23a}$$

$$\mathbb{A}(8;5) \quad : \quad \frac{\mathfrak{so}(8)_x}{\mathfrak{so}(5)_x \oplus \mathfrak{so}(3)_{2x}}, \qquad \mathbb{P} = \mathbb{A}(8;7) \quad : \quad \frac{\mathfrak{so}(8)_x}{\mathfrak{so}(7)_x} \tag{3.23b}$$

$$\mathbb{A}(10;5) \quad : \quad \frac{\mathfrak{so}(10)_x}{\mathfrak{so}(5)_x \oplus \mathfrak{so}(5)_x}, \qquad \mathbb{A}(10;7) \quad : \quad \frac{\mathfrak{so}(10)_x}{\mathfrak{so}(7)_x \oplus \mathfrak{so}(3)_{2x}}, \qquad \mathbb{P} = \mathbb{A}(10;9) \quad : \quad \frac{\mathfrak{so}(10)_x}{\mathfrak{so}(9)_x} \tag{3.23c}$$

for $\mathfrak{so}(6)$, $\mathfrak{so}(8)$ and $\mathfrak{so}(10)$. More generally, $\{\mathbb{A}(2n;r)\}$ contains m distinct $\mathbb{Z}_2$-type outer automorphisms for $\mathfrak{so}(4m)$ and $m+1$ for $\mathfrak{so}(4m+2)$, and this list includes *all* the $\mathbb{Z}_2$-type outer automorphisms given in the table for $\mathfrak{so}(2n)$.

Since all the $\mathbb{Z}_2$-type outer automorphisms are inner-automorphically equivalent to the $\mathbb{Z}_2$-Dynkin automorphism, we have

$$\mathbb{C} = \mathbb{A}(2n;n) \simeq \mathbb{A}(2n;n+2) \simeq \cdots \simeq \mathbb{A}(2n;2n-1) = \mathbb{P}, \qquad \text{for } n \text{ odd} \tag{3.24a}$$

$$\mathbb{A}(2n;n+1) \simeq \mathbb{A}(2n;n+3) \simeq \cdots \simeq \mathbb{A}(2n;2n-1) = \mathbb{P}, \qquad \text{for } n \text{ even}\ . \tag{3.24b}$$

We emphasize however that each automorphism $\mathbb{A}(2n,r)$, $r = n, \dots, 2n-1$, inner or outer, gives rise to a physically distinct twisted sector.

We also note that the parity automorphism $\mathbb{P} = \mathbb{A}(2n;2n-1)$ is the only automorphism in $\{\mathbb{A}(2n;r)\}$ which is an outer automorphism of $\mathfrak{so}(2n)$ for all n. Moreover, the charge-conjugation automorphism $\mathbb{C} = \mathbb{A}(2n;n)$ is the only automorphism in $\{\mathbb{A}(2n;r)\}$ for which a full set of Cartan generators (see Eq. (3.15)) is in g/h - and hence the only one which is equivalent to sign reversal of all the weights of each $\mathfrak{spin}(2n)$.

In what follows, we defer to the mathematicians by treating the case of $\mathbb{P} = \mathbb{A}(2n;2n-1)$ separately, but we will often include $\mathbb{C} = \mathbb{A}(2n;n)$ as a special case of the full set $\{\mathbb{A}(2n;r)\}$.

3.4 The triality automorphism $\mathbb{T}_1$

In this subsection we consider the first triality automorphism $\mathbb{T}_1 \in \mathbb{Z}_3$ which, as we shall see, is equivalent to the Dynkin automorphism in entry viii) of the table in App. A. In this discussion we will use the notation

$$\{J_{ij}(z)\}, \qquad 1 \le i < j \le 8 \quad ; \quad \{J_{\mu\nu}(z)\}, \qquad 1 \le \mu < \nu \le 7 \tag{3.25}$$

for the currents of $\mathfrak{so}(8)$ and $\mathfrak{so}(7)$ respectively.

We recall first the sequence of maximal subalgebras

$$\mathfrak{so}(8)_x \supset \mathfrak{so}(7)_x \supset (\mathfrak{g}_2)_x \tag{3.26}$$

each with embedding index one. Refs. [19, 20] include discussions of this embedding, but we can work out what we need from the fact (see e.g. Ref. [21]) that $\mathfrak{g}_2$ is the subalgebra of $\mathfrak{so}(7)$ which leaves invariant the octonionic structure constants.

For the octonions $\{1, i_\mu\}$ we will use the basis

$$i_\mu i_\nu = -\delta_{\mu\nu} + g_{\mu\nu\rho} i_\rho, \qquad \mu, \nu = 1 \dots 7 \tag{3.27a}$$

$$g_{123} = g_{247} = g_{451} = g_{562} = g_{634} = g_{375} = g_{716} = 1, \qquad g_{\alpha\rho\sigma} g_{\beta\rho\sigma} = 6\delta_{\alpha\beta} \tag{3.27b}$$

and then we need to solve for the 14-dimensional subspace of $\mathfrak{g}_2$ currents $\{J_A(z)\}$

$$J_A(z) = (\rho_A)_{\mu\nu} J_{\mu\nu}(z), \qquad A = 1 \dots 14 \tag{3.28a}$$

$$(\rho_A)_{\alpha\alpha'} g_{\alpha'\beta\gamma} + (\rho_A)_{\beta\beta'} g_{\alpha\beta'\gamma} + (\rho_A)_{\gamma\gamma'} g_{\alpha\beta\gamma'} = 0, \qquad (\rho_A)_{\nu\mu} = -(\rho_A)_{\mu\nu} \ . \tag{3.28b}$$

An explicit form of the fourteen ρ_A's in a trace-orthogonal basis

$$\mathrm{Tr}(\rho_A \rho_B) = -\frac{1}{2}\delta_{AB}, \qquad g_{\alpha\mu\nu}(\rho_A)_{\mu\nu} = 0, \qquad \forall\ A, \alpha \tag{3.29}$$

is given in App. C. The ρ's are proportional to the 7×7 matrix representation $T^{(7)}$ of the 7 of $\mathfrak{g}_2$

$$T_A^{(7)} \equiv 2i\rho_A, \qquad [T_A^{(7)}, T_B^{(7)}] = i f_{ABC} T_C^{(7)} \tag{3.30a}$$

$$f_{ABC} \equiv -4\mathrm{Tr}([\rho_A, \rho_B]\rho_C) \tag{3.30b}$$

where f_{ABC} are the structure constants of $\mathfrak{g}_2$. The second part of (3.29) tells us that the remaining seven currents in $\mathfrak{so}(7)$ can be taken as

$$J_\alpha(z) \equiv g_{\alpha\mu\nu} J_{\mu\nu}(z), \qquad \alpha = 1 \dots 7 \ . \tag{3.31}$$

These and the other seven currents $J_{\alpha 8}(z)$, $\alpha = 1 \ldots 7$ of $\mathfrak{so}(8)$ transform as 7's of $\mathfrak{g}_2$.

It is convenient to introduce the following linear combinations of the two 7's

$$J_\alpha^\pm(z) \equiv \frac{1}{\sqrt{2}}(J_{\alpha 8}(z) \pm \frac{i}{2\sqrt{3}} J_\alpha(z)), \qquad \alpha = 1 \ldots 7 \tag{3.32}$$

which also transform as 7's under $\mathfrak{g}_2$. Then the $\mathfrak{so}(8)$ current algebra takes the form

$$J_A(z) J_B(w) = \frac{k\delta_{AB}}{(z-w)^2} + \frac{i f_{ABC} J_C(w)}{z-w} + \mathcal{O}(z-w)^0 \tag{3.33a}$$

$$J_A(z) J_\alpha^\pm(w) = -\frac{(T_A^{(7)})_{\alpha\beta} J_\beta^\pm(w)}{z-w} + \mathcal{O}(z-w)^0 = \frac{i f_{A\alpha\beta} J_\beta^\pm(w)}{z-w} + \mathcal{O}(z-w)^0 \tag{3.33b}$$

$$J_\alpha^\pm(z) J_\beta^\pm(w) = \pm\sqrt{\frac{2}{3}} \frac{g_{\alpha\beta\gamma} J_\gamma^\mp(w)}{z-w} + \mathcal{O}(z-w)^0 \tag{3.33c}$$

$$J_\alpha^\pm(z) J_\beta^\mp(w) = \frac{k\delta_{\alpha\beta}}{(z-w)^2} + \frac{i f_{\alpha\beta A} J_A(w)}{z-w} + \mathcal{O}(z-w)^0 \tag{3.33d}$$

where $g_{\alpha\beta\gamma}$ are the octonionic structure constants in (3.27b), f_{ABC} are the $\mathfrak{g}_2$ structure constants in (3.30b) and

$$f_{A\alpha\beta} = f_{\alpha\beta A} = -2(\rho_A)_{\alpha\beta} = i(T_A^{(7)})_{\alpha\beta} \ . \tag{3.34}$$

App. C collects some steps and identities used in deriving Eq. (3.33).

The result (3.33) is the desired diagonal basis for the action of the triality automorphism $\mathbb{T}_1$:

$$\omega_{A,B} = \delta_{AB}, \qquad \omega_{\alpha\pm,\beta\pm} = \delta_{\alpha,\beta} e^{\mp\frac{2\pi i}{3}} \tag{3.35a}$$

$$J_A(z)' = J_A(z), \qquad J_\alpha^\pm(z)' = e^{\mp\frac{2\pi i}{3}} J_\alpha^\pm(z) \ . \tag{3.35b}$$

$$A = 1 \ldots 14, \qquad \alpha = 1 \ldots 7 \ . \tag{3.35c}$$

In agreement with entry viii) of the table in App. A, we see that the triality automorphism $\mathbb{T}_1$ defines the coset space

$$\frac{g}{h} = \frac{\mathfrak{so}(8)_x}{(\mathfrak{g}_2)_x} \tag{3.36}$$

which is not a symmetric space.

3.5 The triality automorphism $\mathbb{T}_2$

We turn finally to the second triality automorphism $\mathbb{T}_2$ (see entry ix) of the table in App. A) which is inner-automorphically equivalent to the triality automorphism $\mathbb{T}_1$ but defines the inequivalent coset space

$$\frac{g}{h} = \frac{\mathfrak{so}(8)_x}{\mathfrak{su}(3)_{3x}} \tag{3.37}$$

with invariant subalgebra $h = \mathfrak{su}(3)$ instead of $h = \mathfrak{g}_2$ for $\mathbb{T}_1$. In this discussion, we shall use the standard (Gell-Mann) Cartesian basis

$$\eta_{AB} = \delta_{AB}, \qquad f_{AB}{}^{C} = f_{ABC}, \qquad A, B, C = 1 \ldots 8 \tag{3.38}$$

to describe the $\mathfrak{su}(3)$ and, in this basis, any matrix irrep T of $\mathfrak{su}(3)$ satisfies

$$T_A^\dagger = T_A, \qquad [T_A, T_B] = i f_{ABC} T_C \tag{3.39a}$$

$$\bar{T}_A = -T_A^{\mathrm{t}}, \qquad \bar{T}_A^\dagger = \bar{T}_A, \qquad [\bar{T}_A, \bar{T}_B] = i f_{ABC} \bar{T}_C \tag{3.39b}$$

where t is matrix transpose. Note that we have introduced two equivalent labellings for the same indices $A, B, i, j = 1 \ldots 8$ where i, j are also the vector indices of the generators J_{ij} of $\mathfrak{so}(8)$.

We consider first the $\mathfrak{su}(3)$ currents as they are embedded in the affine $\mathfrak{so}(8)$. These may be taken as

$$J_A(z) \equiv -\frac{i}{2}(T_A^{\mathrm{adj}})_{ij} J_{ij}(z) = -\frac{1}{2} f_{Aij} J_{ij}(z) = -\frac{1}{2} f_{ABC} J_{BC}(z) \tag{3.40}$$

where $\{T_A^{\mathrm{adj}}\}$ is the adjoint rep of $\mathfrak{su}(3)$, and we find from (3.1a) that these currents satisfy

$$J_A(z) J_B(w) = \frac{k_{\mathrm{eff}} \delta_{AB}}{(z-w)^2} + \frac{i f_{ABC} J_C(w)}{z-w} + \mathcal{O}(z-w)^0 \tag{3.41a}$$

$$k_{\mathrm{eff}} \equiv k \frac{Q_{\mathfrak{su}(3)}}{2}, \qquad Q_{\mathfrak{su}(3)} \delta_{AB} = f_{ACD} f_{BCD} \ . \tag{3.41b}$$

Then we may compute the invariant level $x_{\mathfrak{su}(3)}$ of the $\mathfrak{su}(3)$ currents

$$x_{\mathfrak{su}(3)} = \frac{2k_{\mathrm{eff}}}{\psi^2_{\mathfrak{su}(3)}} = k \tilde{h}_{\mathfrak{su}(3)} = 3k = 3x \quad \Rightarrow \quad \psi^2_{\mathfrak{su}(3)} = \frac{2}{3}, \qquad Q_{\mathfrak{su}(3)} = 2, \qquad k_{\mathrm{eff}} = k \tag{3.42}$$

in agreement with the embedding shown in Eq. (3.37).

The rest of the $\mathfrak{so}(8)$ currents transform as the $\mathbf{10}$ and $\overline{\mathbf{10}}$ of $h = \mathfrak{su}(3)$. We find the explicit form of these currents as

$$J^{\pm}_{IJK}(z) = \frac{1}{4} e^{\pm \frac{\pi i}{6}} (g^{\pm}_{IJK})_{ij} J_{ij}(z) \tag{3.43a}$$

$$(g^+_{IJK})_{AB} \equiv (T^{(3)}_A)_{IL}(T^{(3)}_B)_{JM}\epsilon_{LMK} + 5 \text{ terms}, \qquad \epsilon_{123} = 1 \tag{3.43b}$$

$$(g^-_{IJK})_{AB} \equiv (g^+_{IJK})^*_{AB} = (\bar{T}^{(3)}_A)_{IL}(\bar{T}^{(3)}_B)_{JM}\epsilon_{LMK} + 5 \text{ terms} \tag{3.43c}$$

$$\mathrm{Tr}(T^{\mathrm{adj}}_A g^\pm_{IJK}) = \mathrm{Tr}(g^\pm_{IJK} g^\pm_{LMN}) = 0 \tag{3.43d}$$

$$A, B, i, j = 1 \ldots 8 \quad ; \qquad I, J, K, L, M, N = 1 \ldots 3 \quad ; \qquad \{IJK\} = 1 \ldots 10 \tag{3.43e}$$

where ϵ_{LMK} is the Levi-Civita density and the indices IJK of the tensors $g^\pm_{IJK}$ are completely symmetrized as indicated. The matrix irreps $T^{(3)}$ and $\bar{T}^{(3)}$ are the $\mathfrak{3}$ and $\bar{\mathfrak{3}}$ of $\mathfrak{su}(3)$.

In what follows, we will generally use the composite notation

$$\alpha = \{IJK\} = 1 \ldots 10 \tag{3.44}$$

for the symmetrized indices. Then we find the following simple form of the desired diagonal basis for $\mathbb{T}_2$:

$$J_A(z)J_B(w) = \frac{k\delta_{AB}}{(z-w)^2} + \frac{if_{ABC}J_C(w)}{z-w} + \mathcal{O}(z-w)^0 \tag{3.45a}$$

$$J_A(z)J^+_\alpha(w) = -\frac{(T^{(\mathfrak{10})}_A)_{\alpha\beta}J^+_\beta(w)}{z-w} + \mathcal{O}(z-w)^0 \tag{3.45b}$$

$$J_A(z)J^-_\alpha(w) = -\frac{(\bar{T}^{(\mathfrak{10})}_A)_{\alpha\beta}J^-_\beta(w)}{z-w} + \mathcal{O}(z-w)^0 \tag{3.45c}$$

$$J^+_\alpha(z)J^-_\beta(w) = \frac{k(\mathbb{1})_{\alpha\beta}}{(z-w)^2} - \frac{(T^{(\mathfrak{10})}_A)_{\alpha\beta}J_A(w)}{z-w} + \mathcal{O}(z-w)^0 \tag{3.45d}$$

$$J^-_\alpha(z)J^+_\beta(w) = \frac{k(\mathbb{1})_{\alpha\beta}}{(z-w)^2} - \frac{(\bar{T}^{(\mathfrak{10})}_A)_{\alpha\beta}J_A(w)}{z-w} + \mathcal{O}(z-w)^0 \tag{3.45e}$$

$$J^\pm_\alpha(z)J^\mp_\beta(w) = \pm\frac{\tilde{g}_{\alpha\beta\gamma}J^\pm_\gamma(w)}{z-w} + \mathcal{O}(z-w)^0 \ . \tag{3.45f}$$

Here we have defined the 10×10 matrices

$$(T^{(\mathfrak{10})}_A)_{IJK,LMN} \equiv \frac{1}{12}\left\{(T^{(3)}_A)_{IL}\delta_{JM}\delta_{KN} + 35 \text{ terms}\right\} \tag{3.46a}$$

$$(\bar{T}_A^{(\mathbf{10})})_{IJK,LMN} \equiv -(T_A^{(\mathbf{10})})_{LMN,IJK} = \frac{1}{12}\left\{(\bar{T}_A^{(3)})_{IL}\delta_{JM}\delta_{KN} + 35 \text{ terms}\right\} \tag{3.46b}$$

which are the $\mathbf{10}$ and $\overline{\mathbf{10}}$ irreps of $\mathfrak{su}(3)$, and the additional tensors

$$\tilde{g}_{IJK,LMN,PQR} \equiv \frac{1}{216}(\epsilon_{ILP}\epsilon_{JMQ}\epsilon_{KNR} + 215 \text{ terms}) \tag{3.47a}$$

$$(\mathbb{1})_{IJK,LMN} \equiv \frac{1}{36}(\delta_{IL}\delta_{JM}\delta_{KN} + 35 \text{ terms}) \ . \tag{3.47b}$$

Useful identities among all these matrices and tensors are collected in App. D, where it is also noted that the matrix $\mathbb{1}$ in (3.47b) is the natural unit matrix in the 10×10 space.

In the basis (3.45), the action of the triality automorphism $\mathbb{T}_2$ has the same diagonal form

$$J_A(z)' = J_A(z), \qquad J_\alpha^\pm(z)' = e^{\mp\frac{2\pi i}{3}} J_\alpha^\pm(z) \tag{3.48a}$$

$$A = 1\ldots 8, \qquad \alpha = 1\ldots 10 \tag{3.48b}$$

as that given for $\mathbb{T}_1$ in (3.35).

3.6 The affine-Sugawara constructions

In the Cartesian basis (3.1), the affine-Sugawara construction [6, 7, 22, 16, 23] for level $k = x$ of $\mathfrak{so}(2n)$ is well-known

$$T(z) = \frac{1}{2k+Q} :\sum_{i<j} J_{ij}(z)J_{ij}(z):, \qquad Q = 4(n-1), \qquad c = \frac{xn(2n-1)}{x+2(n-1)} \tag{3.49}$$

and this construction is easily rewritten in the diagonal bases above: In any basis, the general form of the affine-Sugawara construction on simple g is

$$T(z) = L_g^{ab} : J_a(z)J_b(z):, \qquad L_g^{ab} = \frac{\eta^{ab}}{2k+Q_g} \tag{3.50a}$$

$$J_a(z)J_b(w) = \frac{k\eta_{ab}}{(z-w)^2} + \frac{if_{ab}{}^c J_c(w)}{z-w} + \mathcal{O}(z-w)^0, \qquad a,b = 1\ldots \dim g \tag{3.50b}$$

so that the Killing metric η_{ab} and its inverse η^{ab} are easily read from the current-current OPEs in any basis.

We find in particular for the diagonal bases of each of our examples:

$\mathbb{P}$

$$T(z) = \frac{1}{2k+Q} :\sum_{\mu<\nu} J_{\mu\nu}(z)J_{\mu\nu}(z) + \sum_\mu J_\mu(z)J_\mu(z): \tag{3.51}$$

$\{\mathbb{A}(2n;r)\}$

$$T(z) = \frac{1}{2k+Q} : \sum_{A<B} J_{AB}(z)J_{AB}(z) + \sum_{I<J} J_{IJ}(z)J_{IJ}(z) + \sum_{A,I} J_{AI}(z)J_{AI}(z) : \tag{3.52}$$

$\mathbb{T}_1$ and $\mathbb{T}_2$

$$T(z) = \frac{1}{2k+Q} : \sum_{A} J_A(z)J_A(z) + \sum_{\alpha} \left(J_\alpha^+(z)J_\alpha^-(z) + J_\alpha^-(z)J_\alpha^+(z)\right) : \ . \tag{3.53}$$

The form (3.53) for the triality automorphisms can also be checked directly using the sum rules in (C.6) and (D.5a). It is easy to check that each affine-Sugawara construction in this list is invariant

$$T(z)' = L_g^{ab} : J_a(z)' J_b(z)' : = T(z) \tag{3.54}$$

under the diagonal action of each of the automorphisms discussed above.

4 The Twisted Sectors $\frac{\mathfrak{so}(2n)}{\mathbb{P}}$, $\left\{\frac{\mathfrak{so}(2n)}{\mathbb{A}(2n;r)}\right\}$ and $\frac{\mathfrak{so}(8)}{\mathbb{T}_1}$, $\frac{\mathfrak{so}(8)}{\mathbb{T}_2}$

4.1 The twisted current algebras

In current-algebraic orbifold theory [3, 4, 5, 12, 13, 15, 14], the twisted OPEs and monodromies of the twisted sectors of each orbifold $A(H)/H$ are obtained from the OPEs and automorphic responses of the untwisted sector $A(H)$ by the method of eigenfields and the principle of local isomorphisms [1, 3, 5, 12]. As a simple example, the monodromy of the general twisted left-mover currents $\hat{J}$ of sector σ is

$$\hat{J}_{n(r)\mu}(ze^{2\pi i}, \sigma) = E_{n(r)}(\sigma)\hat{J}_{n(r)\mu}(z, \sigma), \qquad E_{n(r)}(\sigma) = e^{-2\pi i \frac{n(r)}{\rho(\sigma)}} \tag{4.1a}$$

$$\hat{J}_{n(r)\pm\rho(\sigma),\mu}(z, \sigma) = \hat{J}_{n(r)\mu}(z, \sigma) \tag{4.1b}$$

$$\bar{n}(r) \in \{0, \ldots, \rho(\sigma) - 1\}, \qquad \sigma = 0, \ldots, N_c - 1 \tag{4.1c}$$

where $\rho(\sigma)$ is the order $h_\sigma \in H$, N_c is the number of conjugacy classes of H and the spectral indices $\{n(r)\}$ are determined from the H-eigenvalue problem [3, 5, 12]

$$\omega(h_\sigma)U^\dagger(\sigma)^{n(r)\mu} = U^\dagger(\sigma)^{n(r)\mu} E_{n(r)}(\sigma) \ . \tag{4.2}$$

Here $\omega(h_\sigma)$ is the action of $h_\sigma \in H$ on the untwisted currents, $\{\mu\}$ are the degeneracy indices of the eigenvalue problem and the integers $\bar{n}(r)$ in (4.1c) are the pullback of the spectral indices to the fundamental range.

Solving the H-eigenvalue problem is equivalent to finding a diagonal basis for the automorphism, and since we have already found such diagonal bases, we may now choose the trivial solution

$$U^\dagger(\sigma) = 1\!\!1, \qquad E_{n(r)} = \text{diag}\,\omega(h_\sigma) \tag{4.3}$$

from which the spectral indices are easily deduced. Then we obtain in particular

$\frac{\mathfrak{so}(2n)}{\mathbb{P}}$

$$\rho = 2, \qquad \bar{n}_{\mu\nu} = 0, \qquad \bar{n}_\mu = 1 \tag{4.4a}$$

$$\hat{J}_{0,\mu\nu}(ze^{2\pi i}) = \hat{J}_{0,\mu\nu}(z), \qquad \hat{J}_{1,\mu}(ze^{2\pi i}) = -\hat{J}_{1,\mu}(z) \tag{4.4b}$$

$\left\{\frac{\mathfrak{so}(2n)}{\mathbb{A}(2n;r)}\right\}$

$$\rho = 2, \qquad \bar{n}_{AB} = \bar{n}_{IJ} = 0, \qquad \bar{n}_{AI} = 1 \tag{4.5a}$$

$$\hat{J}_{0,AB}(ze^{2\pi i}) = \hat{J}_{0,AB}(z), \qquad \hat{J}_{0,IJ}(ze^{2\pi i}) = \hat{J}_{0,IJ}(z), \qquad \hat{J}_{1,AI}(ze^{2\pi i}) = -\hat{J}_{1,AI}(z) \tag{4.5b}$$

$\frac{\mathfrak{so}(8)}{\mathbb{T}_1}$ and $\frac{\mathfrak{so}(8)}{\mathbb{T}_2}$

$$\rho = 3, \qquad \bar{n}_A = 0, \qquad \bar{n}_{+1,\alpha} = 1, \qquad \bar{n}_{-1,\alpha} = 2 \tag{4.6a}$$

$$\hat{J}_{0,A}(ze^{2\pi i}) = \hat{J}_{0,A}(z), \qquad \hat{J}_{\pm 1,\alpha}(ze^{2\pi i}) = e^{\mp\frac{2\pi i}{3}}\hat{J}_{\pm 1,\alpha}(z) \tag{4.6b}$$

for the twisted sectors $\mathfrak{so}(2n)/H$ of the orbifolds on $\mathfrak{so}(2n)$.

Returning to the general case, the monodromy (4.1a) gives the expansion

$$\hat{J}_{n(r)\mu}(z,\sigma) = \sum_{m\in\mathbb{Z}} \hat{J}_{n(r)\mu}(m + \tfrac{n(r)}{\rho(\sigma)}) z^{-(m+\frac{n(r)}{\rho(\sigma)})-1} \tag{4.7a}$$

$$\hat{J}_{n(r)\pm\rho(\sigma)}(m + \tfrac{n(r)\pm\rho(\sigma)}{\rho(\sigma)}) = \hat{J}_{n(r)\mu}(m \pm 1 + \tfrac{n(r)}{\rho(\sigma)}) \tag{4.7b}$$

where $\hat{J}_{n(r)\mu}(m + \frac{n(r)}{\rho(\sigma)})$ are the general twisted current modes. This gives in particular for our examples:

$\frac{\mathfrak{so}(2n)}{\mathbb{P}}$

$$\hat{J}_{0,\mu\nu}(z) = \sum_m \hat{J}_{0,\mu\nu}(m) z^{-m-1}, \qquad \hat{J}_{1,\mu}(z) = \sum_m \hat{J}_{1,\mu}(m + \tfrac{1}{2}) z^{-(m+\frac{1}{2})-1} \tag{4.8}$$

$\left\{\frac{\mathfrak{so}(2n)}{\mathbb{A}(2n;r)}\right\}$

$$\hat{J}_{0,AB}(z) = \sum_m \hat{J}_{0,AB}(m) z^{-m-1}, \qquad \hat{J}_{0,IJ}(z) = \sum_m \hat{J}_{0,IJ}(m) z^{-m-1} \tag{4.9a}$$

$$\hat{J}_{1,AI}(z) = \sum_m \hat{J}_{1,AI}(m+\tfrac{1}{2}) z^{-(m+\frac{1}{2})-1} \tag{4.9b}$$

$\frac{\mathfrak{so}(8)}{\mathbb{T}_1}$ and $\frac{\mathfrak{so}(8)}{\mathbb{T}_2}$

$$\hat{J}_{0,A}(z) = \sum_m \hat{J}_{0,A}(m) z^{-m-1}, \qquad \hat{J}_{\pm 1,\alpha}(z) = \sum_m \hat{J}_{\pm 1,\alpha}(m \pm \tfrac{1}{3}) z^{-(m\pm\frac{1}{3})-1} \ . \tag{4.10}$$

We also give the examples of mode periodicity

$$\hat{J}_{2,AB}(m+\tfrac{2}{2}) = \hat{J}_{0,AB}(m+1), \quad \hat{J}_{2,IJ}(m+\tfrac{2}{2}) = \hat{J}_{0,IJ}(m+1), \quad \hat{J}_{-1,AI}(m-\tfrac{1}{2}) = \hat{J}_{1,AI}(m-1+\tfrac{1}{2}) \tag{4.11a}$$

$$\hat{J}_{\pm 2,\alpha}(m \pm \tfrac{2}{3}) = \hat{J}_{\mp 1,\alpha}(m \pm 1 \mp \tfrac{1}{3}) \tag{4.11b}$$

which follow from the general form in Eq. (4.7b).

For any current-algebraic orbifold $A(H)/H$, the general twisted current algebra of sector σ is [5, 12]

$$\begin{aligned} [\hat{J}_{n(r)\mu}(m+\tfrac{n(r)}{\rho(\sigma)}), \hat{J}_{n(s)\nu}(n+\tfrac{n(s)}{\rho(\sigma)})] &= i\mathcal{F}_{n(r)\mu,n(s)\nu}{}^{n(r)+n(s),\delta}(\sigma) \hat{J}_{n(r)+n(s),\delta}(m+n+\tfrac{n(r)+n(s)}{\rho(\sigma)}) \\ &\quad + (m+\tfrac{n(r)}{\rho(\sigma)}) \delta_{m+n+\frac{n(r)+n(s)}{\rho(\sigma)},0} \mathcal{G}_{n(r)\mu;-n(r),\nu}(\sigma) \ . \end{aligned} \tag{4.12a}$$

$$\sigma = 0, \ldots, N_c - 1 \tag{4.12b}$$

where the zero modes $\{\hat{J}_{0\mu}(0)\}$ generate the residual symmetry algebra and Ward identities of sector σ. General formulas are given for the twisted structure constants $\mathcal{F}(\sigma)$ and the twisted metric $\mathcal{G}(\sigma)$ in Refs. [3, 5], but these reduce to $\mathcal{G} = k\eta$ and $\mathcal{F} = f$ for trivial normalization $\chi = 1$ and the trivial solution (4.3) of the H-eigenvalue problem.

This gives in particular the outer-automorphically twisted current algebra for the twisted sector $\frac{\mathfrak{so}(2n)}{\mathbb{P}}$:

$$\begin{aligned} [\hat{J}_{0,\mu\nu}(m), \hat{J}_{0,\rho\sigma}(n)] &= i(\delta_{\nu\rho}\hat{J}_{0,\mu\sigma}(m+n) + \delta_{\mu\sigma}\hat{J}_{0,\nu\rho}(m+n) \\ &\quad - \delta_{\mu\rho}\hat{J}_{0,\nu\sigma}(m+n) - \delta_{\nu\sigma}\hat{J}_{0,\mu\rho}(m+n)) \\ &\quad + 2k(e_{\mu\nu})_{\rho\sigma} m \delta_{m+n,0} \end{aligned} \tag{4.13a}$$

$$[\hat{J}_{0,\mu\nu}(m), \hat{J}_{1,\rho}(n+\tfrac{1}{2})] = i(\delta_{\nu\rho}\hat{J}_{1,\mu}(m+n+\tfrac{1}{2}) - \delta_{\mu\sigma}\hat{J}_{1,\nu}(m+n+\tfrac{1}{2})) \tag{4.13b}$$

$$[\hat{J}_{1,\mu}(m+\tfrac{1}{2}), \hat{J}_{1,\nu}(n+\tfrac{1}{2})] = -i\hat{J}_{0,\mu\nu}(m+n+1) + k\delta_{\mu\nu}(m+\tfrac{1}{2})\delta_{m+n+1,0} \tag{4.13c}$$

$$\hat{J}_{0,\mu\nu}(m)^\dagger = \hat{J}_{0,\mu\nu}(-m), \qquad \hat{J}_{1,\mu}(m+\tfrac{1}{2})^\dagger = \hat{J}_{-1,\mu}(-m-\tfrac{1}{2}) = \hat{J}_{1,\mu}(-m-1+\tfrac{1}{2}) \tag{4.13d}$$

$$\{\mu,\nu,\rho,\sigma\} = \{1,\dots,2n-1\} \tag{4.13e}$$

and for the twisted sectors $\left\{\frac{\mathfrak{so}(2n)}{\mathbb{A}(2n;r)}\right\}$:

$$\begin{aligned}[\hat{J}_{0,AB}(m),\hat{J}_{0,CD}(n)] &= i(\delta_{BC}\hat{J}_{0,AD}(m+n)+\delta_{AD}\hat{J}_{0,BC}(m+n)\\ &\quad -\delta_{AC}\hat{J}_{0,BD}(m+n)-\delta_{BD}\hat{J}_{0,AC}(m+n))\\ &\quad +2k(e_{AB})_{CD}m\delta_{m+n,0}\end{aligned} \tag{4.14a}$$

$$\begin{aligned}[\hat{J}_{0,IJ}(m),\hat{J}_{0,KL}(n)] &= i(\delta_{JK}\hat{J}_{0,IL}(m+n)+\delta_{IL}\hat{J}_{0,JK}(m+n)\\ &\quad -\delta_{IK}\hat{J}_{0,JL}(m+n)-\delta_{JL}\hat{J}_{0,IK}(m+n))\\ &\quad +2k(e_{IJ})_{KL}m\delta_{m+n,0}\end{aligned} \tag{4.14b}$$

$$[\hat{J}_{0,AB}(m),\hat{J}_{0,IJ}(n)] = 0 \tag{4.14c}$$

$$\begin{aligned}[\hat{J}_{1,AI}(m+\tfrac{1}{2}),\hat{J}_{1,BJ}(n+\tfrac{1}{2})] &= -i(\delta_{AB}\hat{J}_{0,IJ}(m+n+1)+\delta_{IJ}\hat{J}_{0,AB}(m+n+1))\\ &\quad +k\delta_{AB}\delta_{CD}(m+\tfrac{1}{2})\delta_{m+n+1,0}\end{aligned} \tag{4.14d}$$

$$[\hat{J}_{0,AB}(m),\hat{J}_{1,CI}(n+\tfrac{1}{2})] = i(\delta_{BC}\hat{J}_{1,AI}(m+n+\tfrac{1}{2})-\delta_{AC}\hat{J}_{1,BI}(m+n+\tfrac{1}{2}))$$

$$[\hat{J}_{0,IJ}(m),\hat{J}_{1,AK}(n+\tfrac{1}{2})] = i(\delta_{JK}\hat{J}_{1,AI}(m+n+\tfrac{1}{2})-\delta_{IK}\hat{J}_{1,AJ}(m+n+\tfrac{1}{2})) \tag{4.14e}$$

$$\hat{J}_{0,AB}(m)^\dagger = \hat{J}_{0,AB}(-m), \qquad \hat{J}_{0,IJ}(m)^\dagger = \hat{J}_{0,IJ}(-m) \tag{4.14f}$$

$$\hat{J}_{1,AI}(m+\tfrac{1}{2})^\dagger = \hat{J}_{-1,AI}(-m-\tfrac{1}{2}) = \hat{J}_{1,AI}(-m-1+\tfrac{1}{2}) \tag{4.14g}$$

$$\{A,B,C,D\} = \{1\dots r\}, \qquad \{I,J,K,L\} = \{r+1,\dots,2n\}, \qquad r = n,\dots,2n-1 \tag{4.14h}$$

and for the twisted triality sector $\frac{\mathfrak{so}(8)}{\mathbb{T}_1}$:

$$[\hat{J}_{0,A}(m),\hat{J}_{0,B}(n)] = if_{ABC}\hat{J}_{0,C}(m+n)+k\delta_{AB}m\delta_{m+n,0} \tag{4.15a}$$

$$[\hat{J}_{0,A}(m),\hat{J}_{\pm1,\alpha}(n\pm\tfrac{1}{3})] = if_{A\alpha\beta}\hat{J}_{\pm1,\beta}(m+n\pm\tfrac{1}{3}) \tag{4.15b}$$

$$[\hat{J}_{\pm1,\alpha}(m \pm \tfrac{1}{3}), \hat{J}_{\pm1,\beta}(n \pm \tfrac{1}{3})] = \pm\sqrt{\frac{2}{3}} g_{\alpha\beta\gamma} \hat{J}_{\pm2,\gamma}(m+n \pm \tfrac{2}{3}) = \pm\sqrt{\frac{2}{3}} g_{\alpha\beta\gamma} \hat{J}_{\mp1,\gamma}(m+n \pm 1 \mp \tfrac{1}{3}) \tag{4.15c}$$

$$[\hat{J}_{\pm1,\alpha}(m \pm \tfrac{1}{3}), \hat{J}_{\mp1,\beta}(n \mp \tfrac{1}{3})] = i f_{\alpha\beta A} \hat{J}_{0,A}(m+n) + k\delta_{\alpha\beta}(m \pm \tfrac{1}{3})\delta_{m+n,0} \tag{4.15d}$$

$$\hat{J}_{0,A}(m)^\dagger = \hat{J}_{0,A}(m), \qquad \hat{J}_{\pm1,\alpha}(m \pm \tfrac{1}{3})^\dagger = \hat{J}_{\mp1,\alpha}(-m \mp \tfrac{1}{3}) \tag{4.15e}$$

$$\{A,B,C\} = \{1 \dots 14\}, \qquad \{\alpha,\beta,\gamma\} = \{1 \dots 7\} \tag{4.15f}$$

and finally for the twisted triality sector $\frac{\mathfrak{so}(8)}{\mathbb{T}_2}$:

$$[\hat{J}_{0,A}(m), \hat{J}_{0,B}(n)] = i f_{ABC} \hat{J}_{0,C}(m+n) + k\delta_{AB} m \delta_{m+n,0} \tag{4.16a}$$

$$[\hat{J}_{0,A}(m), \hat{J}_{+1,\alpha}(n + \tfrac{1}{3})] = -(T_A^{(\mathbf{10})})_{\alpha\beta} \hat{J}_{+1,\beta}(m+n+\tfrac{1}{3}) \tag{4.16b}$$

$$[\hat{J}_{0,A}(m), \hat{J}_{-1,\alpha}(n - \tfrac{1}{3})] = -(\bar{T}_A^{(\mathbf{10})})_{\alpha\beta} \hat{J}_{-1,\beta}(m+n-\tfrac{1}{3}) \tag{4.16c}$$

$$[\hat{J}_{+1,\alpha}(m + \tfrac{1}{3}), \hat{J}_{-1,\beta}(n - \tfrac{1}{3})] = -(T_A^{(\mathbf{10})})_{\alpha\beta} \hat{J}_{0,A}(m+n) + k(\mathbb{1})_{\alpha\beta}(m + \tfrac{1}{3})\delta_{m+n,0} \tag{4.16d}$$

$$[\hat{J}_{-1,\alpha}(m - \tfrac{1}{3}), \hat{J}_{+1,\beta}(n + \tfrac{1}{3})] = -(\bar{T}_A^{(\mathbf{10})})_{\alpha\beta} \hat{J}_{0,A}(m+n) + k(\mathbb{1})_{\alpha\beta}(m - \tfrac{1}{3})\delta_{m+n,0} \tag{4.16e}$$

$$[\hat{J}_{\pm1,\alpha}(m \pm \tfrac{1}{3}), \hat{J}_{\pm1,\beta}(n \pm \tfrac{1}{3})] = \pm\tilde{g}_{\alpha\beta\gamma} \hat{J}_{\mp1,\gamma}(m+n \pm 1 \mp \tfrac{1}{3}) \tag{4.16f}$$

$$\hat{J}_{0,A}(m)^\dagger = \hat{J}_{0,A}(m), \qquad \hat{J}_{\pm1,\alpha}(m \pm \tfrac{1}{3})^\dagger = \hat{J}_{\mp1,\alpha}(-m \mp \tfrac{1}{3}) \tag{4.16g}$$

$$\{A,B,C\} = \{1 \dots 8\}, \qquad \{\alpha,\beta,\gamma\} = \{1 \dots 10\}\ . \tag{4.16h}$$

The twisted current algebras of the sectors $\mathfrak{so}(8)/\mathbb{T}_1^2$ and $\mathfrak{so}(8)/\mathbb{T}_2^2$ are discussed in Subsec. 5.2.

4.2 Rectification

In current-algebraic orbifold theory, it is known [12] that the twisted right-mover current algebra

$$\begin{aligned} [\hat{\bar{J}}_{n(r)\mu}(m + \tfrac{n(r)}{\rho(\sigma)}), \hat{\bar{J}}_{n(s)\nu}(n + \tfrac{n(s)}{\rho(\sigma)})] \;&=\; i\mathcal{F}_{n(r)\mu,n(s)\nu}{}^{n(r)+n(s),\delta}(\sigma)\hat{\bar{J}}_{n(r)+n(s),\delta}(m+n+\tfrac{n(r)+n(s)}{\rho(\sigma)}) \\ &\quad -(m + \tfrac{n(r)}{\rho(\sigma)})\delta_{m+n+\frac{n(r)+n(s)}{\rho(\sigma)},0}\mathcal{G}_{n(r)\mu;-n(r),\nu}(\sigma) \end{aligned} \tag{4.17}$$

of sector $\sigma \leftrightarrow h_\sigma \in H$ is the same as the twisted left-mover current algebra (4.12), but with the *sign reversal* of the central term shown here. As discussed in Ref. [12], the twisted right-mover current algebra (4.17) is isomorphic to the twisted left-mover current algebra of sector h_σ^{-1} and, as a consequence, the form (4.17) is in agreement[‡3] with earlier analysis at the level of characters [24].

Nevertheless, it has been found on a case-by-case basis that each of the twisted right-mover current algebras so far considered can be *rectified* [12] by a linear transformation into a copy $\hat{\bar{J}}^\sharp$ of the twisted left-mover current algebra:

- the WZW permutation orbifolds [12, 13, 14]
- the inner-automorphic WZW orbifolds [12] on simple g
- the (outer-automorphic) charge conjugation orbifold on $\mathfrak{su}(n \geq 3)$ [13].

So far as the basic types of twisted right-mover current algebras are concerned, this leaves the rectification problem open only for the other outer-automorphically twisted affine Lie algebras on simple g.

As a simple example of rectification, we consider first the general inner- or outer-automorphic WZW orbifold of $\mathbb{Z}_2$-type, whose twisted left-mover current algebra has the general form

$$[\hat{J}_{0,A}(m), \hat{J}_{0,B}(n)] = i\mathcal{F}_{0,A;0,B}{}^{0,C}\hat{J}_{0,C}(m+n) + \mathcal{G}_{0,A;0,B}m\delta_{m+n,0} \tag{4.18a}$$

$$[\hat{J}_{0,A}(m), \hat{J}_{1,I}(n+\tfrac{1}{2})] = i\mathcal{F}_{0,A;1,I}{}^{1,J}\hat{J}_{1,J}(m+n+\tfrac{1}{2}) \tag{4.18b}$$

$$[\hat{J}_{1,I}(m+\tfrac{1}{2}), \hat{J}_{1,J}(n+\tfrac{1}{2})] = i\mathcal{F}_{1,I;1,J}{}^{0,A}\hat{J}_{0,A}(m+n+1) + \mathcal{G}_{1,I;1,J}(m+\tfrac{1}{2})\delta_{m+n+1,0} \tag{4.18c}$$

$$A, B, C \in h, \qquad I, J \in g/h \tag{4.18d}$$

with g/h a symmetric space. This form includes in particular the $\mathbb{Z}_2$-twisted current algebras of all the special cases

$$\left\{\frac{\mathfrak{so}(2n)}{\mathbb{A}(2n;r)}\right\} \supset \frac{\mathfrak{so}(2n)}{\mathbb{C}} \quad \dots \quad \frac{\mathfrak{so}(2n)}{\mathbb{P}} \tag{4.19}$$

given above. Then, reversing the signs of the central terms to obtain the corresponding twisted right-mover current algebra, we find that the following redefinition

$$\hat{\bar{J}}^\sharp_{0,A}(m) \equiv \hat{\bar{J}}_{0,A}(-m), \qquad \hat{\bar{J}}^\sharp_{1,I}(m+\tfrac{1}{2}) \equiv \hat{\bar{J}}_{-1,I}(-m-\tfrac{1}{2}) = \hat{\bar{J}}_{1,I}(-m-1+\tfrac{1}{2}) \tag{4.20}$$

rectifies the twisted right-mover current algebra into a copy of the twisted left-mover current algebra (4.18). In fact, there is a theorem [12] which tells us that the twisted right-mover currents of all $\mathbb{Z}_2$-type orbifolds are rectifiable because $h_\sigma^{-1} = h_\sigma$ for the non-trivial element of the $\mathbb{Z}_2$. The result (4.20) shows explicitly that this can be done without extra phases.

[‡3]The bracket analogue of the twisted right-mover current algebra (4.17) also follows from the general WZW orbifold action [12].

On the other hand, we find from Eq. (4.15) that some non-trivial phases are necessary

$$\hat{\bar{J}}^{\sharp}_{0,A}(m) = \hat{\bar{J}}_{0,A}(-m), \qquad \hat{\bar{J}}^{\sharp}_{\pm1,\alpha}(m \pm \tfrac{1}{3}) = -\hat{\bar{J}}_{\mp1,\alpha}(-m \mp \tfrac{1}{3}) \tag{4.21}$$

to rectify the twisted right-mover current algebra of the twisted triality sector $\mathfrak{so}(8)/\mathbb{T}_1$.

We have also been able to find the rectification for the twisted triality sector $\mathfrak{so}(8)/\mathbb{T}_2$:

$$\hat{\bar{J}}^{\sharp}_{0,A}(m) = \omega_{AB}\hat{\bar{J}}_{0,B}(-m), \qquad \hat{\bar{J}}^{\sharp}_{\pm1,\alpha}(m \pm \tfrac{1}{3}) = -\hat{\bar{J}}_{\mp1,\alpha}(-m \mp \tfrac{1}{3}) \ . \tag{4.22}$$

In this case ω_{AB} cannot be trivial and the simplest choice is $\omega = \omega(\mathbb{C})$, the (outer-automorphic) action of charge conjugation $\mathbb{C}$ on $\mathfrak{su}(3)$:

$$T_A^{(3)\prime} = \omega_{AB}T_B^{(3)} = \bar{T}_A^{(3)}, \qquad \bar{T}_A^{(3)\prime} = \omega_{AB}\bar{T}_B^{(3)} = T_A^{(3)} \tag{4.23a}$$

$$\Rightarrow \quad T_A^{(\mathbf{10})\prime} = \omega_{AB}T_B^{(\mathbf{10})} = \bar{T}_A^{(\mathbf{10})}, \qquad \bar{T}_A^{(\mathbf{10})\prime} = \omega_{AB}\bar{T}_B^{(\mathbf{10})} = T_A^{(\mathbf{10})} \ . \tag{4.23b}$$

The diagonal action of this automorphism in this Cartesian basis is given in Ref. [13], where it is noted that the symmetric space

$$\frac{g}{h} = \frac{\mathfrak{su}(3)_{3x}}{\mathfrak{so}(3)_{12x}} \tag{4.24}$$

is defined by this automorphism. As discussed in Subsec. 5.2, the rectifications (4.21) and (4.22) hold as well for the twisted triality sectors $\mathfrak{so}(8)/\mathbb{T}_1^2$ and $\mathfrak{so}(8)/\mathbb{T}_2^2$ respectively.

Together with the conclusions of Refs. [12, 13, 14], this completes the rectification of all the basic types of twisted right-mover current algebras. The rectification problem is not completely solved however because there exist more general twisted current algebras associated to the composition of automorphisms of different basic types. Examples of these are the “doubly-twisted” current algebras of Refs. [2, 4], which result from the composition of permutations of copies of $\mathfrak{g}$ with inner automorphisms of $\mathfrak{g}$.

4.3 The twisted affine-Sugawara constructions

For any current-algebraic orbifold $A(H)/H$, the *twisted affine-Virasoro construction* of sector σ is [3, 5]

$$\hat{T}_\sigma(z) = \mathcal{L}^{n(r)\mu;-n(r)\nu}(\sigma) : \hat{J}_{n(r)\mu}(z,\sigma)\hat{J}_{-n(r)\nu}(z,\sigma) :, \qquad \hat{c} = c \tag{4.25}$$

where $\mathcal{L}(\sigma)$ is the twisted inverse inertia tensor and $: \cdot :$ is operator product normal ordering [2, 3, 5] of the twisted currents. The explicit form [3, 5] of $\mathcal{L}(\sigma) = \mathcal{L}(L,\sigma)$ for all $A(H)/H$ is a duality transformation of the inverse inertia tensor L^{ab} of the corresponding untwisted affine-Virasoro construction [9, 25, 10, 11] of the symmetric CFT $A(H)$. The special case of the WZW orbifolds is described by the general *twisted affine-Sugawara construction* [3, 5]

$$\hat{T}_\sigma(z) = \mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}{}^{n(r)\mu;-n(r)\nu}(\sigma) : \hat{J}_{n(r)\mu}(z,\sigma)\hat{J}_{-n(r)\nu}(z,\sigma) :, \qquad \hat{c}_g = c_g \tag{4.26a}$$

$$\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}(\sigma) = \mathcal{L}(L_g, \sigma) \tag{4.26b}$$

where L_g^{ab} is the ordinary inverse inertia tensor of the untwisted affine-Sugawara construction [6, 7, 22, 16, 23] $T = L_g^{ab} : J_a J_b :$ in the symmetric theory $A_g(H)$.

Because each of our automorphisms acts in a diagonal basis, the twisted inverse inertia tensor of each of our twisted sectors is equal to the ordinary inverse inertia tensor $\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}(\sigma) = L_g$. Then Eqs. (3.51)-(3.53) and (4.26) give the explicit forms of the twisted affine-Sugawara constructions

$\frac{\mathfrak{so}(2n)}{\mathbb{P}}$

$$\hat{T}(z) = \frac{1}{2k+Q} : \sum_{\mu<\nu} \hat{J}_{0,\mu\nu}(z)\hat{J}_{0,\mu\nu}(z) + \sum_{\mu} \hat{J}_{1,\mu}(z)\hat{J}_{-1,\mu}(z) : \tag{4.27}$$

$\left\{\frac{\mathfrak{so}(2n)}{\mathbb{A}(2n;r)}\right\}$

$$\hat{T}(z) = \frac{1}{2k+Q} : \sum_{A<B} \hat{J}_{0,AB}(z)\hat{J}_{0,AB}(z) + \sum_{I<J} \hat{J}_{0,IJ}(z)\hat{J}_{0,IJ}(z) + \sum_{A,I} \hat{J}_{1,AI}(z)\hat{J}_{-1,AI}(z) : \tag{4.28}$$

$\frac{\mathfrak{so}(8)}{\mathbb{T}_1}$ and $\frac{\mathfrak{so}(8)}{\mathbb{T}_2}$

$$\hat{T}(z) = \frac{1}{2k+Q} : \sum_{A} \hat{J}_{0,A}(z)\hat{J}_{0,A}(z) + \sum_{\alpha} (\hat{J}_{1,\alpha}(z)\hat{J}_{-1,\alpha}(z) + \hat{J}_{-1,\alpha}(z)\hat{J}_{1,\alpha}(z)) : \tag{4.29}$$

for each of our examples.

We turn next to the left- and right-mover conformal weights $\hat{\Delta}_0(\sigma)$, $\hat{\bar{\Delta}}_0(\sigma)$ of the scalar twist-field state of sector σ, whose general form is

$$\hat{T}_\sigma(z) = \sum_m L_\sigma(m) z^{-m-2}, \qquad L_\sigma(m \geq 0)|0\rangle_\sigma = \delta_{m,0}\hat{\Delta}_0(\sigma)|0\rangle_\sigma \tag{4.30a}$$

$$\hat{\Delta}_0(\sigma) = \sum_{r,\mu,\nu} \mathcal{L}^{n(r)\mu;-n(r),\nu}(\sigma) \frac{\bar{n}(r)}{2\rho(\sigma)} \left(1 - \frac{\bar{n}(r)}{\rho(\sigma)}\right) \mathcal{G}_{n(r)\mu;-n(r),\nu}(\sigma) = \hat{\bar{\Delta}}_0(\sigma) \tag{4.30b}$$

in each sector of every current-algebraic orbifold $A(H)/H$. The result (4.30b) is most easily derived by going over to the M or mode-ordered form [5, 12] of the Virasoro generators and using (2.1c), (2.1d). The special case of (4.30b) for the WZW orbifolds

$$\hat{\Delta}_0(\sigma) = \sum_{r,\mu,\nu} \mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)\mu;-n(r),\nu}(\sigma) \frac{\bar{n}(r)}{2\rho(\sigma)} \left(1 - \frac{\bar{n}(r)}{\rho(\sigma)}\right) \mathcal{G}_{n(r)\mu;-n(r),\nu}(\sigma) \tag{4.31}$$

is not difficult to evaluate for particular classes of WZW orbifolds (see Refs. [12, 13, 14] for the WZW permutation orbifolds, Ref. [13] for the outer-automorphic charge conjugation orbifold on $\mathfrak{su}(n)$ and Ref. [14] for the inner-automorphic WZW orbifolds).

For our purposes, we note that the general result (4.31) is also easy to evaluate for all low-order inner or outer automorphisms of the large class of permutation-invariant Lie algebras g

$$g = \oplus \mathfrak{g}^I, \qquad \mathfrak{g}^I \simeq \mathfrak{g}, \qquad k_I = k \tag{4.32a}$$

$$\hat{\Delta}_0(\sigma) = \epsilon \dim g/h \frac{x_{\mathfrak{g}}}{x_{\mathfrak{g}} + \tilde{h}_{\mathfrak{g}}}, \qquad \epsilon = \begin{cases} \frac{1}{16} & \text{for } \rho = 2 \\ \frac{1}{18} & \text{for } \rho = 3 \ . \end{cases} \tag{4.32b}$$

Here h is the invariant subalgebra of g, while $\tilde{h}_{\mathfrak{g}}$ and $x_{\mathfrak{g}}$ are respectively the dual Coxeter number of $\mathfrak{g}$ and the invariant level of affine $\mathfrak{g}$. To see this for order $\rho = 2$ and simple g, use the identities

$$\mathcal{G}_{0,A;0,B} = k\eta_{0,A;0,B}, \qquad \eta^{0,A;0,B}\eta_{0,A;0,B} = \dim h \tag{4.33a}$$

$$\mathcal{G}_{1,I;1,J} = k\eta_{1,I;1,J}, \qquad \eta^{1,I;1,J}\eta_{1,I;1,J} = \dim g/h \tag{4.33b}$$

and similarly for $\rho = 3$ and for the permutation-invariant algebras in (4.32a).

Then Eq. (4.32b) on $g = \mathfrak{so}(2n)$ gives the twist-field conformal weights for our $\mathbb{Z}_2$ examples

$\frac{\mathfrak{so}(2n)}{\mathbb{P}}$

$$\hat{J}_{0,\mu\nu}(m \geq 0)|0\rangle_\sigma = \hat{J}_{1,\mu}(m + \tfrac{1}{2} \geq 0)|0\rangle_\sigma = 0 \tag{4.34a}$$

$$\hat{\Delta}_0\left(\frac{\mathfrak{so}(2n)}{\mathbb{P}}\right) = \frac{(2n-1)x}{16(x + 2(n-1))} \tag{4.34b}$$

$\left\{\frac{\mathfrak{so}(2n)}{\mathbb{A}(2n;r)}\right\}$

$$\hat{J}_{0,AB}(m \geq 0)|0\rangle_\sigma = \hat{J}_{0,IJ}(m \geq 0)|0\rangle_\sigma = \hat{J}_{1,AI}(m + \tfrac{1}{2} \geq 0)|0\rangle_\sigma = 0 \tag{4.35a}$$

$$\hat{\Delta}_0\left(\frac{\mathfrak{so}(2n)}{\mathbb{A}(2n;r)}\right) = \frac{r(2n-r)}{16}\frac{x}{x + 2(n-1)}, \qquad r = n, \ldots, 2n-1 \tag{4.35b}$$

$$\frac{\mathfrak{so}(2n)}{\mathbb{C}} = \frac{\mathfrak{so}(2n)}{\mathbb{A}(2n;n)} : \qquad \hat{\Delta}_0\left(\frac{\mathfrak{so}(2n)}{\mathbb{C}}\right) = \frac{n^2 x}{16(x + 2(n-1))} \tag{4.35c}$$

where x is the invariant level of the affine $\mathfrak{so}(2n)$.

For the twisted triality sectors on $g = \mathfrak{so}(8)$, Eq. (4.32b) gives

$\frac{\mathfrak{so}(8)}{\mathbb{T}_1}$ and $\frac{\mathfrak{so}(8)}{\mathbb{T}_2}$

$$\hat{J}_{0,A}(m \geq 0)|0\rangle_\sigma = \hat{J}_{\pm 1,\alpha}(m \pm \tfrac{1}{3} \geq 0)|0\rangle_\sigma = 0 \tag{4.36a}$$

$$\hat{\Delta}_0\left(\frac{\mathfrak{so}(8)}{\mathbb{T}_1}\right) = \frac{7}{9}\,\frac{x}{x+6}, \qquad \hat{\Delta}_0\left(\frac{\mathfrak{so}(8)}{\mathbb{T}_2}\right) = \frac{10}{9}\,\frac{x}{x+6} \tag{4.36b}$$

where x is the invariant level of the affine $\mathfrak{so}(8)$. Because $\mathbb{T}_1^2$ and $\mathbb{T}_2^2$ also have order $\rho = 3$, Eq. (4.32b) tells us that the conformal weights of the scalar twist-fields of the twisted sectors $\mathfrak{so}(8)/\mathbb{T}_1^2$ and $\mathfrak{so}(8)/\mathbb{T}_2^2$

$$\hat{\Delta}_0\left(\frac{\mathfrak{so}(8)}{\mathbb{T}_1^2}\right) = \hat{\Delta}_0\left(\frac{\mathfrak{so}(8)}{\mathbb{T}_1}\right), \qquad \hat{\Delta}_0\left(\frac{\mathfrak{so}(8)}{\mathbb{T}_2^2}\right) = \hat{\Delta}_0\left(\frac{\mathfrak{so}(8)}{\mathbb{T}_2}\right) \tag{4.37}$$

are the same as those given in (4.36b). The twisted sectors $\mathfrak{so}(8)/\mathbb{T}_1^2$ and $\mathfrak{so}(8)/\mathbb{T}_2^2$ are further discussed in Subsec. 5.2.

Note that the scalar twist-field conformal weights are different for each of the $\mathbb{Z}_2$-type twisted sectors on $\mathfrak{so}(2n)$ and for the twisted triality sectors $\mathfrak{so}(8)/\mathbb{T}_1$ and $\mathfrak{so}(8)/\mathbb{T}_2$, even though all the outer automorphisms on each g are inner-automorphically equivalent. This is not surprising since it is well known that inner-automorphic spectral flow [6, 26] affects conformal weights.

For completeness, we also evaluate (4.32b) to give the scalar twist-field conformal weights of the twisted sectors of all the other outer-automorphic orbifolds on simple g:

$$\hat{\Delta}_0\left(\frac{\mathfrak{su}(n)}{\mathfrak{so}(n)}\right) = \frac{(n-1)(n+2)}{32}\frac{x}{x+n} \tag{4.38a}$$

$$\hat{\Delta}_0\left(\frac{\mathfrak{su}(2n)}{\mathfrak{sp}(n)}\right) = \frac{(n-1)(2n+1)}{16}\frac{x}{x+2n} \tag{4.38b}$$

$$\hat{\Delta}_0\left(\frac{E_6}{F_4}\right) = \frac{13}{8}\frac{x}{x+12}, \qquad \hat{\Delta}_0\left(\frac{E_6}{C_4}\right) = \frac{21}{8}\frac{x}{x+8}\ . \tag{4.38c}$$

Here we have chosen to label the twisted sectors by their corresponding symmetric spaces g/h, and x is the invariant level of affine g. For the charge conjugation orbifold on $\mathfrak{su}(n)$, the result (4.38a) was given earlier in Ref. [13].

Taken with the corresponding results given in Refs. [12, 13, 14], this completes the computation of the scalar twist-field conformal weights in each sector of all the basic WZW orbifolds.

4.4 The twisted KZ systems

In the general theory of WZW orbifolds [3, 5, 12, 13, 14] the description is extended to include the *twisted affine primary fields* $\hat{g} = \hat{g}_-\hat{g}_+$ of sector σ and their OPEs, e.g.

$$\hat{J}_{n(r)\mu}(z,\sigma)\hat{g}_+(\mathcal{T},w,\sigma) = \frac{\hat{g}_+(\mathcal{T},w,\sigma)}{z-w}\mathcal{T}_{n(r)\mu}(T,\sigma) + \mathcal{O}(z-w)^0 \tag{4.39}$$

where $\hat{g}_+(\mathcal{T},z,\sigma)$ is the left-mover twisted affine primary field in twisted representation $\mathcal{T} \equiv \mathcal{T}(T,\sigma)$. When acting on the scalar twist-field state $|0\rangle_\sigma$ of sector σ, the twisted affine primary fields create the *twisted affine primary states* [15] of sector σ

$$|\mathcal{T}\rangle_\sigma = \lim_{z\to 0}\ \hat{g}_+(\mathcal{T},z,\sigma)z^{\gamma(\mathcal{T},\sigma)}|0\rangle_\sigma \tag{4.40a}$$

$$\hat{J}_{n(r)\mu}(m+\tfrac{n(r)}{\rho(\sigma)}\geq 0)|\mathcal{T}\rangle_\sigma = \delta_{m+\frac{n(r)}{\rho(\sigma)},0}|\mathcal{T}\rangle_\sigma \mathcal{T}_{n(r)\mu}(T,\sigma) \tag{4.40b}$$

where $\gamma(\mathcal{T},\sigma)$ is the so-called matrix exponent of the twisted affine primary field. Moreover, the OPEs of the twisted affine primary fields lead to the *twisted vertex operator equations* of sector σ and the general left- and right-mover *twisted KZ systems* [12, 13, 15] of the WZW orbifolds.

For twisted sector σ of any WZW orbifold, the general twisted left-mover KZ system [12, 13, 14] has the form

$$\hat{A}_+(\mathcal{T},z,\sigma) \equiv {}_\sigma\langle 0|\hat{g}_+(\mathcal{T}^{(1)},z_1,\sigma)\hat{g}_+(\mathcal{T}^{(2)},z_2,\sigma)\cdots\hat{g}_+(\mathcal{T}^{(N)},z_N,\sigma)|0\rangle_\sigma \tag{4.41a}$$

$$\partial_\kappa \hat{A}_+(\mathcal{T},z,\sigma) = \hat{A}_+(\mathcal{T},z,\sigma)\hat{W}_\kappa(\mathcal{T},z,\sigma), \qquad \kappa = 1\ldots N, \qquad \sigma = 0,\ldots,N_c-1 \tag{4.41b}$$

$$\hat{A}_+(\mathcal{T},z,\sigma)\left(\sum_{\rho=1}^{N}\mathcal{T}_{0\mu}^{(\rho)}(T,\sigma)\right) = 0, \qquad \forall\ \mu \tag{4.41c}$$

where the general twisted left-mover connection $\hat{W}_\kappa(\mathcal{T},z,\sigma)$ is given in Eq. (2.4b). The general Ward identities in (4.41c) are associated to the residual symmetry of each sector σ.

Using the data above, the general twisted left-mover KZ system is easily evaluated for each of our examples:

$\frac{\mathfrak{so}(2n)}{\mathbb{P}}$

$$\hat{W}_\kappa(\mathcal{T},z) = \frac{2}{2k+Q}\left[\sum_{\rho\neq\kappa}\frac{1}{z_{\kappa\rho}}\left(\sum_{\mu<\nu}\mathcal{T}_{0,\mu\nu}^{(\rho)}\mathcal{T}_{0,\mu\nu}^{(\kappa)} + \left(\frac{z_\rho}{z_\kappa}\right)^{\frac{1}{2}}\sum_\mu \mathcal{T}_{1,\mu}^{(\rho)}\mathcal{T}_{-1,\mu}^{(\kappa)}\right) - \frac{1}{2z_\kappa}\sum_\mu \mathcal{T}_{1,\mu}^{(\kappa)}\mathcal{T}_{-1,\mu}^{(\kappa)}\right] \tag{4.42a}$$

$$\hat{A}_+(\mathcal{T},z,)\left(\sum_{\kappa=1}^{N}\mathcal{T}_{0,\mu\nu}^{(\kappa)}\right) = 0, \qquad \forall\ \mu\nu \in \mathfrak{so}(2n-1) \tag{4.42b}$$

$\left\{\frac{\mathfrak{so}(2n)}{\mathbb{A}(2n;r)}\right\}$

$$\begin{aligned}\hat{W}_\kappa(\mathcal{T},z) \;=\; & \frac{2}{2k+Q}\left[\sum_{\rho\neq\kappa}\frac{1}{z_{\kappa\rho}}\left(\sum_{A<B}\mathcal{T}_{0,AB}^{(\rho)}\mathcal{T}_{0,AB}^{(\kappa)} + \sum_{I<J}\mathcal{T}_{0,IJ}^{(\rho)}\mathcal{T}_{0,IJ}^{(\kappa)} + \left(\frac{z_\rho}{z_\kappa}\right)^{\frac{1}{2}}\sum_{AI}\mathcal{T}_{1,AI}^{(\rho)}\mathcal{T}_{-1,AI}^{(\kappa)}\right)\right. \\ & \left. -\frac{1}{2z_\kappa}\sum_{AI}\mathcal{T}_{1,AI}^{(\kappa)}\mathcal{T}_{-1,AI}^{(\kappa)}\right]\end{aligned} \tag{4.43a}$$

$$\hat{A}_+(\mathcal{T},z)\left(\sum_{\kappa=1}^{N}\mathcal{T}_{0,AB}^{(\kappa)}\right) = \hat{A}_+(\mathcal{T},z)\left(\sum_{\kappa=1}^{N}\mathcal{T}_{0,IJ}^{(\kappa)}\right) = 0, \qquad \forall\ AB,\ IJ \in \mathfrak{so}(r)\oplus\mathfrak{so}(2n-r) \tag{4.43b}$$

$\frac{\mathfrak{so}(8)}{\mathbb{T}_1}$

$$\hat{W}_\kappa(\mathcal{T},z) = \frac{2}{2k+Q}\left[\sum_{\rho\neq\kappa}\frac{1}{z_{\kappa\rho}}\left(\sum_{A=1}^{14}\mathcal{T}^{(\rho)}_{0,A}\mathcal{T}^{(\kappa)}_{0,A}+\sum_{\alpha=1}^{7}\left(\left(\frac{z_\rho}{z_\kappa}\right)^{\frac{1}{3}}\mathcal{T}^{(\rho)}_{1,\alpha}\mathcal{T}^{(\kappa)}_{-1,\alpha}+\left(\frac{z_\rho}{z_\kappa}\right)^{\frac{2}{3}}\mathcal{T}^{(\rho)}_{-1,\alpha}\mathcal{T}^{(\kappa)}_{1,\alpha}\right)\right)\right.$$
$$\left.-\frac{1}{3z_\kappa}\sum_{\alpha=1}^{7}\left(\mathcal{T}^{(\kappa)}_{1,\alpha}\mathcal{T}^{(\kappa)}_{-1,\alpha}+2\mathcal{T}^{(\kappa)}_{-1,\alpha}\mathcal{T}^{(\kappa)}_{1,\alpha}\right)\right] \tag{4.44a}$$

$$\hat{A}_+(\mathcal{T},z)\left(\sum_{\kappa=1}^{N}\mathcal{T}^{(\kappa)}_{0,A}\right)=0, \qquad \forall\ A\in\mathfrak{g}_2 \tag{4.44b}$$

$\frac{\mathfrak{so}(8)}{\mathbb{T}_2}$

$$\hat{W}_\kappa(\mathcal{T},z) = \frac{2}{2k+Q}\left[\sum_{\rho\neq\kappa}\frac{1}{z_{\kappa\rho}}\left(\sum_{A=1}^{8}\mathcal{T}^{(\rho)}_{0,A}\mathcal{T}^{(\kappa)}_{0,A}+\sum_{\alpha=1}^{10}\left(\left(\frac{z_\rho}{z_\kappa}\right)^{\frac{1}{3}}\mathcal{T}^{(\rho)}_{1,\alpha}\mathcal{T}^{(\kappa)}_{-1,\alpha}+\left(\frac{z_\rho}{z_\kappa}\right)^{\frac{2}{3}}\mathcal{T}^{(\rho)}_{-1,\alpha}\mathcal{T}^{(\kappa)}_{1,\alpha}\right)\right)\right.$$
$$\left.-\frac{1}{3z_\kappa}\sum_{\alpha=1}^{10}\left(\mathcal{T}^{(\kappa)}_{1,\alpha}\mathcal{T}^{(\kappa)}_{-1,\alpha}+2\mathcal{T}^{(\kappa)}_{-1,\alpha}\mathcal{T}^{(\kappa)}_{1,\alpha}\right)\right] \tag{4.45a}$$

$$\hat{A}_+(\mathcal{T},z)\left(\sum_{\kappa=1}^{N}\mathcal{T}^{(\kappa)}_{0,A}\right)=0, \qquad \forall\ A\in\mathfrak{su}(3)\ . \tag{4.45b}$$

The twisted KZ systems of the sectors $\mathfrak{so}(8)/\mathbb{T}_1^2$ and $\mathfrak{so}(8)/\mathbb{T}_2^2$ are discussed in Subsec. 5.2.

The twisted representation matrices $\mathcal{T}=\mathcal{T}(T,\sigma)$ satisfy the general *orbifold Lie algebra*

$$[\mathcal{T}_{n(r)\mu},\mathcal{T}_{n(s)\nu}] = i\mathcal{F}_{n(r)\mu,n(s)\nu}{}^{n(r)+n(s),\delta}(\sigma)\mathcal{T}_{n(r)+n(s),\delta} \tag{4.46a}$$

$$\mathcal{T}_{n(r)\pm\rho(\sigma),\mu}=\mathcal{T}_{n(r)\mu}, \qquad [\mathcal{T}^{(\rho)},\mathcal{T}^{(\kappa)}]=0, \qquad \rho\neq\kappa \tag{4.46b}$$

in sector σ of the general WZW orbifold, where $\mathcal{F}(\sigma)$ are the same twisted structure constants which appear in the general twisted current algebra (4.12). For our examples then, the orbifold Lie algebra takes the specific forms:

$\frac{\mathfrak{so}(2n)}{\mathbb{P}}$

$$[\mathcal{T}_{0,\mu\nu},\mathcal{T}_{0,\rho\sigma}] = i(\delta_{\nu\rho}\mathcal{T}_{0,\mu\sigma}+\delta_{\mu\sigma}\mathcal{T}_{0,\nu\rho}-\delta_{\mu\rho}\mathcal{T}_{0,\nu\sigma}-\delta_{\nu\sigma}\mathcal{T}_{0,\mu\rho}) \tag{4.47a}$$

$$[\mathcal{T}_{0,\mu\nu},\mathcal{T}_{1,\rho}] = i(\delta_{\nu\rho}\mathcal{T}_{1,\mu}-\delta_{\mu\rho}\mathcal{T}_{1,\nu}), \qquad [\mathcal{T}_{1,\mu},\mathcal{T}_{1,\nu}]=-i\mathcal{T}_{2,\mu\nu}=-i\mathcal{T}_{0,\mu\nu} \tag{4.47b}$$

$\left\{\frac{\mathfrak{so}(2n)}{\mathbb{A}(2n;r)}\right\}$

$$[\mathcal{T}_{0,AB},\mathcal{T}_{0,CD}] = i(\delta_{BC}\mathcal{T}_{0,AD}+\delta_{AD}\mathcal{T}_{0,BC}-\delta_{AC}\mathcal{T}_{0,BD}-\delta_{BD}\mathcal{T}_{0,AC}) \tag{4.48a}$$

$$[\mathcal{T}_{0,IJ},\mathcal{T}_{0,KL}]=i(\delta_{JK}\mathcal{T}_{0,IL}+\delta_{IL}\mathcal{T}_{0,JK}-\delta_{IK}\mathcal{T}_{0,JL}-\delta_{JL}\mathcal{T}_{0,IK}) \tag{4.48b}$$

$$[\mathcal{T}_{0,AB},\mathcal{T}_{0,IJ}]=0 \tag{4.48c}$$

$$[\mathcal{T}_{1,AI},\mathcal{T}_{1,BJ}]=-i(\delta_{AB}\mathcal{T}_{0,IJ}+\delta_{IJ}\mathcal{T}_{0,AB}) \tag{4.48d}$$

$$[\mathcal{T}_{0,AB},\mathcal{T}_{1,CI}]=i(\delta_{BC}\mathcal{T}_{1,AI}-\delta_{AC}\mathcal{T}_{1,BI}) \tag{4.48e}$$

$$[\mathcal{T}_{0,IJ},\mathcal{T}_{1,AK}]=i(\delta_{JK}\mathcal{T}_{1,AI}-\delta_{IK}\mathcal{T}_{1,AJ}) \tag{4.48f}$$

$\frac{\mathfrak{so}(8)}{\mathbb{T}_1}$

$$[\mathcal{T}_{0,A},\mathcal{T}_{0,B}]=if_{ABC}\mathcal{T}_{0,C},\qquad [\mathcal{T}_{0,A},\mathcal{T}_{\pm1,\alpha}]=if_{A\alpha\beta}\mathcal{T}_{\pm1,\beta} \tag{4.49a}$$

$$[\mathcal{T}_{\pm1,\alpha},\mathcal{T}_{\pm1,\beta}]=\pm\sqrt{\frac{2}{3}}g_{\alpha\beta\gamma}\mathcal{T}_{\pm2,\gamma}=\pm\sqrt{\frac{2}{3}}g_{\alpha\beta\gamma}\mathcal{T}_{\mp1,\gamma} \tag{4.49b}$$

$$[\mathcal{T}_{\pm1,\alpha},\mathcal{T}_{\mp1,\beta}]=if_{\alpha\beta A}\mathcal{T}_{0,A} \tag{4.49c}$$

$\frac{\mathfrak{so}(8)}{\mathbb{T}_2}$

$$[\mathcal{T}_{0,A},\mathcal{T}_{0,B}]=if_{ABC}\mathcal{T}_{0,C} \tag{4.50a}$$

$$[\mathcal{T}_{0,A},\mathcal{T}_{1,\alpha}]=-(T_A^{(\mathbf{10})})_{\alpha\beta}\mathcal{T}_{+1,\beta},\qquad [\mathcal{T}_{0,A},\mathcal{T}_{-1,\alpha}]=-(\bar{T}_A^{(\mathbf{10})})_{\alpha\beta}\mathcal{T}_{-1,\beta} \tag{4.50b}$$

$$[\mathcal{T}_{+1,\alpha},\mathcal{T}_{-1,\beta}]=-(T_A^{(\mathbf{10})})_{\alpha\beta}\mathcal{T}_{0,A},\qquad [\mathcal{T}_{-1,\alpha},\mathcal{T}_{+1,\beta}]=-(\bar{T}_A^{(\mathbf{10})})_{\alpha\beta}\mathcal{T}_{0,A} \tag{4.50c}$$

$$[\mathcal{T}_{\pm1,\alpha},\mathcal{T}_{\pm1,\beta}]=\pm\tilde{g}_{\alpha\beta\gamma}\mathcal{T}_{\mp1,\gamma}\ . \tag{4.50d}$$

More explicit forms of the twisted representation matrices are discussed in the following subsection.

Using (4.49c), (4.50c) and the fact that $\mathrm{Tr}(T_A^{(\mathbf{10})})=0$, we obtain the simplified form of the triality connection for $\mathfrak{so}(8)/\mathbb{T}_1$ and $\mathfrak{so}(8)/\mathbb{T}_2$

$$\begin{aligned}\hat{W}_\kappa(\mathcal{T},z) \;=\; \frac{2}{2k+Q}\Bigg[&\sum_{\rho\neq\kappa}\frac{1}{z_{\kappa\rho}}\left(\sum_A \mathcal{T}_{0,A}^{(\rho)}\mathcal{T}_{0,A}^{(\kappa)}+\sum_\alpha\left(\left(\frac{z_\rho}{z_\kappa}\right)^{\frac{1}{3}}\mathcal{T}_{1,\alpha}^{(\rho)}\mathcal{T}_{-1,\alpha}^{(\kappa)}+\left(\frac{z_\rho}{z_\kappa}\right)^{\frac{2}{3}}\mathcal{T}_{-1,\alpha}^{(\rho)}\mathcal{T}_{1,\alpha}^{(\kappa)}\right)\right)\\ &-\frac{1}{z_\kappa}\sum_\alpha \mathcal{T}_{1,\alpha}^{(\kappa)}\mathcal{T}_{-1,\alpha}^{(\kappa)}\Bigg]\end{aligned} \tag{4.51}$$

where the ranges of A and α for the two cases are given in (4.44) and (4.45). Although flatness of the twisted connections is guaranteed by the construction, we have used (4.49) and (4.50) to check explicitly and at length that the twisted triality connection (4.51) is not only flat but abelian flat

$$\partial_\rho \hat{W}_\kappa(\mathcal{T}, z) - \partial_\kappa \hat{W}_\rho(\mathcal{T}, z) = [\hat{W}_\kappa(\mathcal{T}, z), \hat{W}_\rho(\mathcal{T}, z)] = 0 \tag{4.52}$$

for both $\mathfrak{so}(8)/\mathbb{T}_1$ and $\mathfrak{so}(8)/\mathbb{T}_2$. As noted in Subsec. 5.2, the twisted KZ connections of the twisted sectors $\mathfrak{so}(8)/\mathbb{T}^2$, $\mathbb{T} = \mathbb{T}_1$ and $\mathbb{T}_2$ are also abelian flat.

Including the explicit check [13] of flatness for all orbifolds of $\mathbb{Z}_2$-type, this completes the explicit check of flatness for the twisted KZ connections of each sector of all the outer-automorphic WZW orbifolds on simple g.

4.5 Representation theory

We turn now to discuss the explicit form of the twisted representation matrices.

For each sector σ of any WZW orbifold $A_g(H)/H$, the general formula for the twisted representation matrices

$$\mathcal{T}_{n(r)\mu} \equiv \mathcal{T}_{n(r)\mu}(T, \sigma) = \chi_{n(r)\mu}(\sigma) U(\sigma)_{n(r)\mu}{}^a U(T, \sigma) T_a U^\dagger(T, \sigma) \tag{4.53}$$

is given in Ref. [12]. Here T_a, $a = 1...\text{dim } g$ can be any untwisted matrix representation of g. The quantities $U^\dagger(\sigma)$ and $U^\dagger(T, \sigma)$ are the eigenvalue matrices of the H-eigenvalue problem and the extended H-eigenvalue problem respectively, while $\{\chi\}$ is a set of normalization constants.

Since we are starting in a diagonal basis for each automorphism, we may set $\chi(\sigma) = U^\dagger(\sigma) = 1$ (see Eq. (4.3)) to obtain a simplified form for the twisted representation matrices

$$\mathcal{T}(T, \sigma) = U(T, \sigma) T U^\dagger(T, \sigma) \ . \tag{4.54}$$

Nevertheless, we must still solve the *linkage relation* [12] for $W(h_\sigma; T)$ given $\omega(h_\sigma)$ and the *extended H-eigenvalue problem* [12] for $U^\dagger(T, \sigma)$

$$W^\dagger(h_\sigma; T) T_a W(h_\sigma; T) = \omega(h_\sigma)_a{}^b T_b \equiv T_a{}' \tag{4.55a}$$

$$W(h_\sigma; T) U^\dagger(T, \sigma) = U^\dagger(T, \sigma) E(T, \sigma) \tag{4.55b}$$

in order to evaluate the twisted representation matrices in (4.54). Here $\omega(h_\sigma)$ is the action of $h_\sigma \in H \subset Aut(g)$ on the untwisted currents of affine g and $W(h_\sigma; T)$ is the action of h_σ in untwisted representation T.

Solution of the relations in (4.55) is straightforward for any "real" irrep T of g, where real is defined here as the unitary equivalence

$$T' = \omega T \cong T \tag{4.56}$$

for any automorphism group H of any g. A simple example is the adjoint representation $T^{\rm adj}$ of any g, for which it is known that [12, 13]

$$(T_a^{\rm adj})_b{}^c = -if_{ab}{}^c, \qquad W(h_\sigma; T^{\rm adj}) = \omega(h_\sigma), \qquad U(T^{\rm adj}, \sigma) = U^\dagger(\sigma) \tag{4.57}$$

in each sector of every WZW orbifold. In our case, this gives the simple form

$$U^\dagger(\sigma) = 1 \qquad \Rightarrow \qquad \mathcal{T}(T^{\rm adj}, \sigma) = T^{\rm adj} \tag{4.58}$$

and the specific results

$$\frac{\mathfrak{so}(2n)}{\mathbb{P}} \quad : \qquad \mathcal{T}_{0,\mu\nu}(T^{\rm adj}) = T^{\rm adj}_{\mu\nu}, \qquad \mathcal{T}_{1,\mu}(T^{\rm adj}) = T^{\rm adj}_{\mu,2n} \tag{4.59}$$

$$\left\{\frac{\mathfrak{so}(2n)}{\mathbb{A}(2n;r)}\right\} \quad : \qquad \mathcal{T}_{0,AB}(T^{\rm adj}) = T^{\rm adj}_{AB}, \qquad \mathcal{T}_{0,IJ}(T^{\rm adj}) = T^{\rm adj}_{IJ}, \qquad \mathcal{T}_{1,AI}(T^{\rm adj}) = T^{\rm adj}_{AI} \tag{4.60}$$

$$\frac{\mathfrak{so}(8)}{\mathbb{T}_1} \quad : \quad \mathcal{T}_{0,A}(T^{\rm adj}) = T^{\rm adj}_A = (\rho_A)_{\mu\nu} T^{\rm adj}_{\mu\nu}, \quad \mathcal{T}_{\pm 1,\alpha}(T^{\rm adj}) = T^{\rm adj\pm}_\alpha = \frac{1}{\sqrt{2}}(T^{\rm adj}_{\alpha 8} \pm \frac{i}{2\sqrt{3}} g_{\alpha\beta\gamma} T^{\rm adj}_{\beta\gamma}) \tag{4.61}$$

$$\frac{\mathfrak{so}(8)}{\mathbb{T}_2} \quad : \quad \mathcal{T}_{0,A}(T^{\rm adj}) = T^{\rm adj}_A = -\frac{1}{2} f_{Aij} T^{\rm adj}_{ij}, \qquad \mathcal{T}_{\pm 1,\alpha}(T^{\rm adj}) = \frac{1}{4} e^{\pm \frac{\pi i}{6}} (g^\pm_\alpha)_{ij} T^{\rm adj}_{ij} \tag{4.62}$$

are obtained for the twisted sectors of this paper. More generally, the solution of the linkage relation (4.55a) is guaranteed for real representations by the unitary equivalence $T' \cong T$.

For outer automorphisms of simple g we must also consider "complex" irreps $T^{\rm (c)}$ for which

$$T^{\rm (c)\prime} = \omega T^{\rm (c)} \ncong T^{\rm (c)} \tag{4.63}$$

and in this case we must take the representation T to be *reducible* [12, 13]. As an example, for any outer automorphism $\omega^2 = 1$ of the $\mathbb{Z}_2$ type it is known that [13]

$$T_a = \begin{pmatrix} T_a^{\rm (c)} & 0 \\ 0 & T_a^{\rm (c)\prime} \end{pmatrix} = \begin{pmatrix} T_a^{\rm (c)} & 0 \\ 0 & \omega_a{}^b T_b^{\rm (c)} \end{pmatrix} \tag{4.64a}$$

$$T_a{}' = \omega_a{}^b T_b = \begin{pmatrix} T_a^{\rm (c)\prime} & 0 \\ 0 & T_a^{\rm (c)} \end{pmatrix} \tag{4.64b}$$

$$W(T) = i \begin{pmatrix} 0 & 1\!\!1 \\ 1\!\!1 & 0 \end{pmatrix}, \qquad W^\dagger(T) W(T) = \begin{pmatrix} 1\!\!1 & 0 \\ 0 & 1\!\!1 \end{pmatrix} \tag{4.64c}$$

$$U(T) = U^\dagger(T) = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{1} & -\mathbb{1} \end{pmatrix}, \qquad U^\dagger(T)U(T) = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \qquad E(T) = i \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \tag{4.64d}$$

where $W(T)$ and $U(T), E(T)$ are respectively the solutions of the linkage relation (4.55a) and the extended H-eigenvalue problem in (4.55b). Then Eq. (4.54) gives in particular for any complex representation

$\frac{\mathfrak{so}(2n)}{\mathbb{P}}$

$$T^{(\mathrm{c})} = (T^{(\mathrm{c})}_{\mu\nu}, T^{(\mathrm{c})}_{\mu,2n}), \qquad T^{(\mathrm{c})\prime} = (T^{(\mathrm{c})}_{\mu\nu}, -T^{(\mathrm{c})}_{\mu,2n}) \tag{4.65a}$$

$$\mathcal{T}_{0,\mu\nu}(T) = U(T)T_{\mu\nu}U^\dagger(T) = \begin{pmatrix} T^{(\mathrm{c})}_{\mu\nu} & 0 \\ 0 & T^{(\mathrm{c})}_{\mu\nu} \end{pmatrix} \tag{4.65b}$$

$$\mathcal{T}_{1,\mu}(T) = U(T)T_{\mu,2n}U^\dagger(T) = \begin{pmatrix} 0 & T^{(\mathrm{c})}_{\mu,2n} \\ T^{(\mathrm{c})}_{\mu,2n} & 0 \end{pmatrix} \tag{4.65c}$$

$\left\{\frac{\mathfrak{so}(2n)}{\mathbb{A}(2n;r)},\ r \text{ odd}\right\}$

$$T^{(\mathrm{c})} = (T^{(\mathrm{c})}_{AB}, T^{(\mathrm{c})}_{IJ}, T^{(\mathrm{c})}_{AI}), \qquad T^{(\mathrm{c})\prime} = (T^{(\mathrm{c})}_{AB}, T^{(\mathrm{c})}_{IJ}, -T^{(\mathrm{c})}_{AI}) \tag{4.66a}$$

$$\mathcal{T}_{0,AB}(T) = U(T)T_{AB}U^\dagger(T) = \begin{pmatrix} T^{(\mathrm{c})}_{AB} & 0 \\ 0 & T^{(\mathrm{c})}_{AB} \end{pmatrix} \tag{4.66b}$$

$$\mathcal{T}_{0,IJ}(T) = U(T)T_{IJ}U^\dagger(T) = \begin{pmatrix} T^{(\mathrm{c})}_{IJ} & 0 \\ 0 & T^{(\mathrm{c})}_{IJ} \end{pmatrix} \tag{4.66c}$$

$$\mathcal{T}_{1,AI}(T) = U(T)T_{AI}U^\dagger(T) = \begin{pmatrix} 0 & T^{(\mathrm{c})}_{AI} \\ T^{(\mathrm{c})}_{AI} & 0 \end{pmatrix} \tag{4.66d}$$

in each of our outer-automorphically twisted sectors of $\mathbb{Z}_2$-type. Specific examples of Eqs. (4.65), (4.66) are obtained by substitution of the Weyl spinor reps $T^{(\mathrm{c})} = S^{(\mathrm{c})} \equiv S$ in App. B. The conjugate Weyl spinors are automatically included as $C^{(\mathrm{c})} \equiv S^{(\mathrm{c})\prime}$ in this formulation.

For either of the triality automorphisms $\mathbb{T}_1$ or $\mathbb{T}_2$ we find instead

$$T_a = \begin{pmatrix} T^{(\mathrm{c})}_a & 0 & 0 \\ 0 & T^{(\mathrm{c})\prime}_a & 0 \\ 0 & 0 & T^{(\mathrm{c})\prime\prime}_a \end{pmatrix} = \begin{pmatrix} T^{(\mathrm{c})}_a & 0 & 0 \\ 0 & \omega_a{}^b T^{(\mathrm{c})}_b & 0 \\ 0 & 0 & (\omega^2)_a{}^b T^{(\mathrm{c})}_b \end{pmatrix} \tag{4.67a}$$

$$T_a{}' = \omega_a{}^b T_b = \begin{pmatrix} T_a^{(\mathrm{c})\prime} & 0 & 0 \\ 0 & T_a^{(\mathrm{c})\prime\prime} & 0 \\ 0 & 0 & T_a^{(\mathrm{c})} \end{pmatrix} \tag{4.67b}$$

$$W(T) = \begin{pmatrix} 0 & 0 & \mathbb{1} \\ \mathbb{1} & 0 & 0 \\ 0 & \mathbb{1} & 0 \end{pmatrix}, \qquad W^\dagger(T)W(T) = W^3(T) = \begin{pmatrix} \mathbb{1} & 0 & 0 \\ 0 & \mathbb{1} & 0 \\ 0 & 0 & \mathbb{1} \end{pmatrix} \tag{4.67c}$$

$$U^\dagger(T) = \frac{1}{\sqrt{3}} \begin{pmatrix} \mathbb{1} & e^{2\pi i/3}\mathbb{1} & e^{-2\pi i/3}\mathbb{1} \\ \mathbb{1} & e^{-2\pi i/3}\mathbb{1} & e^{2\pi i/3}\mathbb{1} \\ \mathbb{1} & \mathbb{1} & \mathbb{1} \end{pmatrix}, \qquad U^\dagger(T)U(T) = \begin{pmatrix} \mathbb{1} & 0 & 0 \\ 0 & \mathbb{1} & 0 \\ 0 & 0 & \mathbb{1} \end{pmatrix} \tag{4.67d}$$

$$E(T) = \begin{pmatrix} \mathbb{1} & 0 & 0 \\ 0 & e^{-2\pi i/3}\mathbb{1} & 0 \\ 0 & 0 & e^{2\pi i/3}\mathbb{1} \end{pmatrix} \tag{4.67e}$$

where $W(T)$ and $U^\dagger(T)$, $E(T)$ are respectively the solutions of the linkage relation (4.55a) and the extended H-eigenvalue problem in (4.55b).

To be more explicit for the twisted triality sectors, we introduce the unified notation

$$T^{(\mathrm{c})} = (T_A^{(\mathrm{c})}, T_\alpha^{(\mathrm{c})+}, T_\alpha^{(\mathrm{c})-}) \tag{4.68a}$$

$$T^{(\mathrm{c})\prime} = (T_A^{(\mathrm{c})}, e^{-\frac{2\pi i}{3}} T_\alpha^{(\mathrm{c})+}, e^{\frac{2\pi i}{3}} T_\alpha^{(\mathrm{c})-}), \qquad T^{(\mathrm{c})\prime\prime} = (T_A^{(\mathrm{c})}, e^{\frac{2\pi i}{3}} T_\alpha^{(\mathrm{c})+}, e^{-\frac{2\pi i}{3}} T_\alpha^{(\mathrm{c})-}) \tag{4.68b}$$

$$\mathbb{T}_1 \quad : \quad T_A^{(\mathrm{c})} = (\rho_A)_{\mu\nu} T_{\mu\nu}^{(\mathrm{c})}, \qquad T_\alpha^{(\mathrm{c})\pm} = \frac{1}{\sqrt{2}} \left(T_{\alpha 8}^{(\mathrm{c})} \pm \frac{i}{2\sqrt{3}} g_{\alpha\beta\gamma} T_{\beta\gamma}^{(\mathrm{c})} \right) \tag{4.68c}$$

$$\mathbb{T}_2 \quad : \quad T_A^{(\mathrm{c})} = -\frac{1}{2} f_{Aij} T_{ij}^{(\mathrm{c})}, \qquad T_\alpha^{(\mathrm{c})\pm} = \frac{1}{4} e^{\pm\frac{\pi i}{6}} (g_\alpha^\pm)_{ij} T_{ij}^{(\mathrm{c})} \tag{4.68d}$$

where $T^{(\mathrm{c})}$ is any complex irrep of $\mathfrak{so}(8) \cong \mathfrak{spin}(8)$ under $\mathbb{T}_1$ or $\mathbb{T}_2$. Then the twisted representation matrices $\mathcal{T}(T)$ of $\mathfrak{so}(8)/\mathbb{T}_1$ or $\mathfrak{so}(8)/\mathbb{T}_2$

$$\mathcal{T}_{0,A}(T) = U(T) T_A U^\dagger(T) = \begin{pmatrix} T_A^{(\mathrm{c})} & 0 & 0 \\ 0 & T_A^{(\mathrm{c})} & 0 \\ 0 & 0 & T_A^{(\mathrm{c})} \end{pmatrix} \tag{4.69a}$$

$$\mathcal{T}_{1,\alpha}(T) = U(T) T_\alpha^+ U^\dagger(T) = e^{\frac{2\pi i}{3}} \begin{pmatrix} 0 & T_\alpha^{(\mathrm{c})+} & 0 \\ 0 & 0 & T_\alpha^{(\mathrm{c})+} \\ T_\alpha^{(\mathrm{c})+} & 0 & 0 \end{pmatrix} \tag{4.69b}$$

$$\mathcal{T}_{-1,\alpha}(T) = U(T)T_\alpha^- U^\dagger(T) = e^{-\frac{2\pi i}{3}} \begin{pmatrix} 0 & 0 & T_\alpha^{(\mathrm{c})-} \\ T_\alpha^{(\mathrm{c})-} & 0 & 0 \\ 0 & T_\alpha^{(\mathrm{c})-} & 0 \end{pmatrix} \tag{4.69c}$$

are obtained from Eqs. (4.54) and (4.67) .

As an explicit example, let us work out the untwisted and twisted representation matrices corresponding to the complex reps $V^{(\mathrm{c})}$, $S^{(\mathrm{c})}$ and $C^{(\mathrm{c})}$ of $\mathfrak{so}(8) \cong \mathfrak{spin}(8)$ under $\mathbb{T}_1$ or $\mathbb{T}_2$. We begin with the familiar form of the untwisted vector rep $V^{(\mathrm{c})}$ of $\mathfrak{so}(8)$ in the standard Cartesian basis:

$$V_{ij}^{(\mathrm{c})} = 2ie_{ij}, \qquad (e_{ij})_{kl} = \frac{1}{2}(\delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il}) \tag{4.70a}$$

$$\mathbb{T}_1 \quad : \quad V_A^{(\mathrm{c})} = 2i(\rho_A)_{\mu\nu}e_{\mu\nu}, \qquad V_\alpha^{(\mathrm{c})\pm} = \frac{2i}{\sqrt{2}}\left(e_{\alpha 8} \pm \frac{i}{2\sqrt{3}} g_{\alpha\beta\gamma}e_{\beta\gamma}\right) \tag{4.70b}$$

$$\mathbb{T}_2 \quad : \quad V_A^{(\mathrm{c})} = -if_{Aij}e_{ij}, \qquad V_\alpha^{(\mathrm{c})\pm} = \frac{i}{2}e^{\pm\frac{\pi i}{6}}(g_\alpha^\pm)_{ij}e_{ij} \ . \tag{4.70c}$$

In this case, we can define both the untwisted Weyl spinor rep and the untwisted conjugate Weyl spinor rep directly from the vector rep and the $\mathbb{Z}_3$ automorphism

$$S^{(\mathrm{c})} = V^{(\mathrm{c})\prime} = \omega V^{(\mathrm{c})} = (V_A^{(\mathrm{c})}, e^{-\frac{2\pi i}{3}}V_\alpha^{(\mathrm{c})+}, e^{\frac{2\pi i}{3}}V_\alpha^{(\mathrm{c})-}) \tag{4.71a}$$

$$C^{(\mathrm{c})} = S^{(\mathrm{c})\prime} = \omega V^{(\mathrm{c})\prime} = \omega^2 V^{(\mathrm{c})} = (V_A^{(\mathrm{c})}, e^{+\frac{2\pi i}{3}}V_\alpha^{(\mathrm{c})+}, e^{-\frac{2\pi i}{3}}V_\alpha^{(\mathrm{c})-}) \tag{4.71b}$$

where ω can be taken as $\omega(\mathbb{T}_1)$ or $\omega(\mathbb{T}_2)$. Then the required reducible representation T is

$$T_a = \begin{pmatrix} V_a^{(\mathrm{c})} & 0 & 0 \\ 0 & (\omega V^{(\mathrm{c})})_a & 0 \\ 0 & 0 & (\omega^2 V^{(\mathrm{c})})_a \end{pmatrix} \tag{4.72}$$

and the twisted representation matrices

$$\mathcal{T}_{0,A}(T) = \begin{pmatrix} V_A^{(\mathrm{c})} & 0 & 0 \\ 0 & V_A^{(\mathrm{c})} & 0 \\ 0 & 0 & V_A^{(\mathrm{c})} \end{pmatrix} \tag{4.73a}$$

$$\mathcal{T}_{+1,\alpha}(T) = e^{\frac{2\pi i}{3}} \begin{pmatrix} 0 & V_\alpha^{(\mathrm{c})+} & 0 \\ 0 & 0 & V_\alpha^{(\mathrm{c})+} \\ V_\alpha^{(\mathrm{c})+} & 0 & 0 \end{pmatrix}, \qquad \mathcal{T}_{-1,\alpha}(T) = e^{-\frac{2\pi i}{3}} \begin{pmatrix} 0 & 0 & V_\alpha^{(\mathrm{c})-} \\ V_\alpha^{(\mathrm{c})-} & 0 & 0 \\ 0 & V_\alpha^{(\mathrm{c})-} & 0 \end{pmatrix} \tag{4.73b}$$

follow from (4.69). For the twisted triality sectors $\mathfrak{so}(8)/\mathbb{T}_1$ and $\mathfrak{so}(8)/\mathbb{T}_2$ the explicit forms of the entries here are given in Eqs. (4.70b,c).

4.6 Action formulation of outer-automorphic WZW orbifolds

The classical theory of WZW orbifolds is described by the *general WZW orbifold action* [12, 13, 15] on the cylinder (ξ,t) and the solid cylinder Γ, which reduces to the form

$$\begin{aligned}\hat{S}[\hat{g}(\mathcal{T},\sigma)] \quad &= \quad -\frac{k}{\epsilon y(T)}\left(\frac{1}{8\pi}\int d^2\xi\ \mathrm{Tr}(\ \hat{g}^{-1}(\mathcal{T},\sigma)\partial_+\hat{g}(\mathcal{T},\sigma)\ \hat{g}^{-1}(\mathcal{T},\sigma)\partial_-\hat{g}(\mathcal{T},\sigma)\ )\right.\\ &\qquad \left. +\frac{1}{12\pi}\int_\Gamma \mathrm{Tr}(\ (\ \hat{g}^{-1}(\mathcal{T},\sigma)d\hat{g}(\mathcal{T},\sigma)\ )^3\ )\right) \end{aligned}\tag{4.74a}$$

$$\mathrm{Tr}(T_aT_b)=y(T)\delta_{ab}, \qquad \epsilon=\begin{cases}1 & \text{for real reps } T\\ 2 & \text{for complex reps } T^{(\mathrm{c})} \text{ when } \rho=2\\ 3 & \text{for complex reps } T^{(\mathrm{c})} \text{ when } \rho=3\end{cases}\tag{4.74b}$$

for each twisted sector σ of all the outer-automorphic WZW orbifolds on simple g. Here $\hat{g}(\mathcal{T},\sigma)\equiv\hat{g}(\mathcal{T}(T,\sigma),\xi,t,\sigma)$ are the *group orbifold elements* of sector σ, which are the high-level or classical limit of the twisted affine primary fields. The group orbifold elements are locally group elements but they exhibit the monodromy

$$\hat{g}(\mathcal{T},\xi+2\pi,t,\sigma)=E(T,\sigma)\hat{g}(\mathcal{T},\xi,t,\sigma)E(T,\sigma)^*\tag{4.75}$$

where $E(T,\sigma)$ is the eigenvalue matrix of the extended H-eigenvalue problem in (4.55b). The result (4.74) generalizes the action given for the charge-conjugation orbifold on $\mathfrak{su}(n)$ in Ref. [13].

The group orbifold elements can be expressed in terms of the twisted tangent space coordinates $\hat{\beta}$

$$\hat{g}(\mathcal{T}(T,\sigma),\xi,t,\sigma)=e^{i\hat{\beta}^{n(r)\mu}(\xi,t)\mathcal{T}_{n(r)\mu}(T,\sigma)}, \qquad \hat{\beta}^{n(r)\mu}(\xi+2\pi,t)=\hat{\beta}^{n(r)\mu}(\xi,t)e^{2\pi i\frac{n(r)}{\rho(\sigma)}}\tag{4.76}$$

where $\mathcal{T}_{n(r)\mu}(T,\sigma)$ are the same twisted representation matrices discussed in the operator formulation above. The consistency of the monodromy of $\hat{\beta}$ in (4.76) and that of $\hat{g}$ in (4.75) is a consequence of a selection rule for the twisted representation matrices [12].

This gives the explicit forms of the group orbifold elements for each of our twisted sectors

$\frac{\mathfrak{so}(2n)}{\mathbb{P}}$

$$\hat{g}(\mathcal{T},\xi)=e^{i(\hat{\beta}^{0,\mu\nu}(\xi)\mathcal{T}_{0,\mu\nu}+\hat{\beta}^{1,\mu}(\xi)\mathcal{T}_{1,\mu})}, \qquad \hat{\beta}^{0,\mu\nu}(\xi+2\pi)=\hat{\beta}^{0,\mu\nu}(\xi), \qquad \hat{\beta}^{1,\mu}(\xi+2\pi)=-\hat{\beta}^{1,\mu}(\xi)\tag{4.77}$$

$\left\{\frac{\mathfrak{so}(2n)}{\mathbb{A}(2n;r)}\right\}$

$$\hat{g}(\mathcal{T},\xi)=e^{i(\hat{\beta}^{0,AB}(\xi)\mathcal{T}_{0,AB}+\hat{\beta}^{0,IJ}(\xi)\mathcal{T}_{0,IJ}+\hat{\beta}^{1,AI}(\xi)\mathcal{T}_{1,AI})}\tag{4.78a}$$

$$\hat{\beta}^{0,AB}(\xi+2\pi)=\hat{\beta}^{0,AB}(\xi),\qquad \hat{\beta}^{0,IJ}(\xi+2\pi)=\hat{\beta}^{0,IJ}(\xi),\qquad \hat{\beta}^{1,AI}(\xi+2\pi)=-\hat{\beta}^{1,AI}(\xi) \tag{4.78b}$$

$\frac{\mathfrak{so}(8)}{\mathbb{T}_1}$ and $\frac{\mathfrak{so}(8)}{\mathbb{T}_2}$

$$\hat{g}(\mathcal{T},\xi)=e^{i(\hat{\beta}^{0,A}(\xi)\mathcal{T}_{0,A}+\hat{\beta}^{\pm1,\alpha}(\xi)\mathcal{T}_{\pm1,\alpha})} \tag{4.79a}$$

$$\hat{\beta}^{0,A}(\xi+2\pi)=\hat{\beta}^{0,A}(\xi),\qquad \hat{\beta}^{\pm1,\alpha}(\xi+2\pi)=\hat{\beta}^{\pm1,\alpha}(\xi)\ e^{\pm\frac{2\pi i}{3}} \tag{4.79b}$$

where $\mathcal{T}=\mathcal{T}(T,\sigma)$ and we have suppressed the time label t. The explicit forms of $\mathcal{T}$ for real and complex representations are discussed in the previous subsection. The corresponding discussion for the charge conjugation orbifold on $\mathfrak{su}(n)$ was given in Ref. [13].

5 Assembling the Orbifolds

5.1 The $\mathbb{Z}_2$ orbifolds on $\mathfrak{so}(2n)$

Using the development above, we may construct the orbifolds

$$\frac{A_{\mathfrak{so}(2n)}\big(\mathbb{Z}_2(\mathbb{P})\big)}{\mathbb{Z}_2(\mathbb{P})},\qquad \frac{A_{\mathfrak{so}(2n)}\big(\mathbb{Z}_2(\mathbb{A}(2n;r))\big)}{\mathbb{Z}_2(\mathbb{A}(2n;r))},\qquad r=n,\dots,2n-3 \tag{5.1}$$

of type $\mathbb{Z}_2$ on $\mathfrak{so}(2n)$. These orbifolds have an untwisted sector $\sigma=0$ and one twisted sector $\sigma=1$, called respectively $\mathfrak{so}(2n)/\mathbb{P}$ and $\mathfrak{so}(2n)/\mathbb{A}(2n;r)$, $r=n,\dots,2n-3$ in this paper. Among these, only those generated by $\mathbb{P}$ and $\{\mathbb{A}(2n;r),\ r=\text{odd}\}$ are outer-automorphic orbifolds, while the others are inner automorphic. Following the counting in Subsec. 3.3, we have then constructed the number N of distinct $\mathbb{Z}_2$-type outer-automorphic orbifolds on $\mathfrak{so}(2n\geq 6)$

$$N=\begin{cases} r & \text{on } \mathfrak{so}(4r)\\ r+1 & \text{on } \mathfrak{so}(4r+2)\ . \end{cases} \tag{5.2}$$

In what follows, we consider the more intricate orbifolds of types $\mathbb{Z}_3$ and S_3 on $\mathfrak{so}(8)$.

5.2 Two $\mathbb{Z}_3$ triality orbifolds on $\mathfrak{so}(8)$

There are two outer-automorphic $\mathbb{Z}_3$ orbifolds on $\mathfrak{so}(8)$

$$\frac{A_{\mathfrak{so}(8)}\big(\mathbb{Z}_3(\mathbb{T}_1)\big)}{\mathbb{Z}_3(\mathbb{T}_1)},\qquad \frac{A_{\mathfrak{so}(8)}\big(\mathbb{Z}_3(\mathbb{T}_2)\big)}{\mathbb{Z}_3(\mathbb{T}_2)} \tag{5.3}$$

and each of these orbifolds has three sectors: the untwisted sector $\sigma=0$, a first twisted sector $\sigma=1$ called $\mathfrak{so}(8)/\mathbb{T}_1$ or $\mathfrak{so}(8)/\mathbb{T}_2$ above, and a second twisted sector $\sigma=2$ which corresponds to $\mathfrak{so}(8)/\mathbb{T}_1^2$ or $\mathfrak{so}(8)/\mathbb{T}_2^2$ respectively. Our task in this subsection is the description of the $\sigma=2$ sectors.

For $\mathbb{T} = \mathbb{T}_1$ or $\mathbb{T}_2$, the action $\omega(\mathbb{T}^2) = \omega(\mathbb{T})^2$ is given by

$$\mathbb{T}^2 \quad : \quad J_A(z)' = J_A(z), \qquad J_\alpha^\pm(z)' = e^{\pm\frac{2\pi i}{3}} J_\alpha^\pm(z) \tag{5.4}$$

in the same diagonal bases given above for $\mathbb{T}_1$ and $\mathbb{T}_2$. The untwisted affine-Sugawara construction (3.53) holds as well for $\mathbb{T}_1^2$ and $\mathbb{T}_2^2$.

The phase reversal for $J_\alpha^\pm(z)'$ in (5.4) relative to the action of $\omega(\mathbb{T})$ tells us that the twisted current algebra of $\mathfrak{so}(8)/\mathbb{T}_1^2$ or $\mathfrak{so}(8)/\mathbb{T}_2^2$ is the same as that given for $\mathfrak{so}(8)/\mathbb{T}_1$ or $\mathfrak{so}(8)/\mathbb{T}_2$ in Eqs. (4.15) and (4.16), but with the map

$$\frac{\mathfrak{so}(8)}{\mathbb{T}} \to \frac{\mathfrak{so}(8)}{\mathbb{T}^2} \quad : \qquad m \pm \tfrac{1}{3} \;\to\; m \mp \tfrac{1}{3}, \qquad \hat{J}_{\pm 1,\alpha}(m \pm \tfrac{1}{3}) \;\to\; \hat{J}_{\mp 1,\alpha}(m \mp \tfrac{1}{3}) \tag{5.5}$$

for both cases. We may then rewrite the twisted current algebras of $\mathfrak{so}(8)/\mathbb{T}^2$ in the standard forms given for $\mathfrak{so}(8)/\mathbb{T}$ in Eqs. (4.15) and (4.16).

For example, we find

$$\frac{\mathfrak{so}(8)}{\mathbb{T}_1^2} \quad : \qquad [\hat{J}_{\pm 1,\alpha}(m \pm \tfrac{1}{3}), \hat{J}_{\pm 1,\beta}(n \pm \tfrac{1}{3})] = \mp\sqrt{\frac{2}{3}} g_{\alpha\beta\gamma} \hat{J}_{\mp 1,\gamma}(m + n \pm 1 \mp \tfrac{1}{3}) \tag{5.6}$$

instead of (4.15c) for $\mathfrak{so}(8)/\mathbb{T}_1$. For the $\mathfrak{so}(8)/\mathbb{T}_1^2$ sector of the $\mathbb{Z}_3(\mathbb{T}_1)$ orbifold, this is the only change in the form of the twisted current algebra.

For the $\mathfrak{so}(8)/\mathbb{T}_2^2$ sector of the $\mathbb{Z}_3(\mathbb{T}_2)$ orbifold, we find again the standard form (4.16), but with the replacement

$$\frac{\mathfrak{so}(8)}{\mathbb{T}_2^2} \quad : \qquad [\hat{J}_{\pm 1,\alpha}(m \pm \tfrac{1}{3}), \hat{J}_{\pm 1,\beta}(n \pm \tfrac{1}{3})] = \mp\tilde{g}_{\alpha\beta\gamma} \hat{J}_{\mp 1,\gamma}(m + n \pm 1 \mp \tfrac{1}{3}) \tag{5.7}$$

and $T^{(\mathbf{10})} \leftrightarrow \bar{T}^{(\mathbf{10})}$ everywhere.

The rectifications (4.21) or (4.22), the twisted affine-Sugawara constructions (4.29) and the scalar twist-field conformal weights (4.37) are the same for the sectors $\mathfrak{so}(8)/\mathbb{T}^2$ as they are for the sectors $\mathfrak{so}(8)/\mathbb{T}$.

The twisted representation matrices $\mathcal{T}(T,\sigma)$ of the $\sigma = 2$ sectors satisfy the same orbifold Lie algebras (4.49) and (4.50) but with the map

$$\frac{\mathfrak{so}(8)}{\mathbb{T}} \to \frac{\mathfrak{so}(8)}{\mathbb{T}^2} \quad : \qquad \mathcal{T}_{\pm 1,\alpha}(T) \;\to\; \mathcal{T}_{\mp 1,\alpha}(T) \tag{5.8}$$

which mirrors the twisted current algebras. This means that we can construct the twisted representation matrices of sector $\sigma = 2$ from the ones discussed above for sector $\sigma = 1$:

$$\mathcal{T}_A(T, \sigma = 2) = \mathcal{T}_A(T, \sigma = 1), \qquad \mathcal{T}_{\pm 1,\alpha}(T, \sigma = 2) = \mathcal{T}_{\mp 1,\alpha}(T, \sigma = 1) \ . \tag{5.9}$$

Then the explicit forms of the twisted KZ connections and Ward identities for the $\sigma = 2$ sectors of the $\mathbb{Z}_3$ triality orbifolds are

$$\hat{W}_\kappa(\mathcal{T}, z, \sigma = 2)$$
$$= \frac{2}{2k+Q}\left[\sum_{\rho\neq\kappa}\frac{1}{z_{\kappa\rho}}\left(\sum_A \mathcal{T}^{(\rho)}_{0,A}\mathcal{T}^{(\kappa)}_{0,A} + \sum_\alpha\left(\left(\frac{z_\rho}{z_\kappa}\right)^{\frac{1}{3}}\mathcal{T}^{(\rho)}_{-1,\alpha}\mathcal{T}^{(\kappa)}_{1,\alpha} + \left(\frac{z_\rho}{z_\kappa}\right)^{\frac{2}{3}}\mathcal{T}^{(\rho)}_{1,\alpha}\mathcal{T}^{(\kappa)}_{-1,\alpha}\right)\right)\right.$$
$$\left. -\frac{1}{z_\kappa}\sum_\alpha \mathcal{T}^{(\kappa)}_{-1,\alpha}\mathcal{T}^{(\kappa)}_{1,\alpha}\right] \tag{5.10a}$$

$$\hat{A}_+(\mathcal{T}, z, \sigma = 2)\left(\sum_{\kappa=1}^N \mathcal{T}^{(\kappa)}_{0,A}\right) = 0 \tag{5.10b}$$

$$A = 1\ldots 14, \qquad \alpha = 1\ldots 7 \qquad \text{for } \frac{\mathfrak{so}(8)}{\mathbb{T}_1^2} \qquad ; \qquad A = 1\ldots 8, \qquad \alpha = 1\ldots 10 \qquad \text{for } \frac{\mathfrak{so}(8)}{\mathbb{T}_2^2} \tag{5.10c}$$

where these twisted representation matrices $\mathcal{T}$, which satisfy (4.49) and (4.50), are the same ones we constructed for $\mathfrak{so}(8)/\mathbb{T}_1$ and $\mathfrak{so}(8)/\mathbb{T}_2$ in Subsec. 4.5. These twisted connections are also abelian flat.

For the group-orbifold elements discussed in Subsec. 4.6, we find instead the forms

$\frac{\mathfrak{so}(8)}{\mathbb{T}_1^2}$ and $\frac{\mathfrak{so}(8)}{\mathbb{T}_2^2}$

$$\hat{g}(\mathcal{T}, \xi + 2\pi, \sigma = 2) = E(T)^* \hat{g}(\mathcal{T}, \xi, \sigma = 2)E(T) \tag{5.11a}$$

$$\hat{g}(\mathcal{T}, \xi, \sigma = 2) = e^{i(\hat{\beta}^{0,A}(\xi)\mathcal{T}_{0,A} + \hat{\beta}^{\pm 1,\alpha}(\xi)\mathcal{T}_{\mp 1,\alpha})} \tag{5.11b}$$

$$\hat{\beta}^{0,A}(\xi + 2\pi) = \hat{\beta}^{0,A}(\xi), \qquad \hat{\beta}^{\pm 1,\alpha}(\xi + 2\pi) = \hat{\beta}^{\pm 1,\alpha}(\xi)\; e^{\pm\frac{2\pi i}{3}} \tag{5.11c}$$

where $E(T)$ and these twisted representation matrices are again the objects defined for the $\sigma = 1$ sectors in Subsec. 4.5.

5.3 Three S_3 triality orbifolds on $\mathfrak{so}(8)$

In the following subsections, we will assemble the three S_3 triality orbifolds on $\mathfrak{so}(8)$:

$$\frac{A_{\mathfrak{so}(8)}(S_3(\mathbb{P}, \mathbb{T}_1))}{S_3(\mathbb{P}, \mathbb{T}_1)}, \quad \frac{A_{\mathfrak{so}(8)}(S_3(\mathbb{A}, \mathbb{T}_2))}{S_3(\mathbb{A}, \mathbb{T}_2)}, \quad \frac{A_{\mathfrak{so}(8)}(S_3(\tilde{\mathbb{A}}, \mathbb{T}_1))}{S_3(\tilde{\mathbb{A}}, \mathbb{T}_1)} \tag{5.12a}$$

$$\mathbb{P} \simeq \mathbb{A}(8;5) \cong \mathbb{A} \cong \tilde{\mathbb{A}}\,. \tag{5.12b}$$

In this notation, each S_3 is generated by the $\mathbb{Z}_2$ and the $\mathbb{Z}_3$ element shown in its argument.

Some comment will be helpful about the variety of $\mathbb{Z}_2$ outer automorphisms in (5.12b). The outer automorphisms $\mathbb{P}$ and $\mathbb{A}(8;5)$, discussed explicitly in Sec. 3, are related by a non-trivial inner automorphism K

$$\mathbb{P} \;:\; \frac{\mathfrak{so}(8)_x}{\mathfrak{so}(7)_x} \qquad ; \qquad \mathbb{A}(8;5) \;:\; \frac{\mathfrak{so}(8)_x}{\mathfrak{so}(5)_x \oplus \mathfrak{so}(3)_{2x}} \tag{5.13a}$$

$$\mathbb{P} \simeq \mathbb{A}(8;5) : \quad \omega(\mathbb{P}) = K\omega(\mathbb{A}(8;5)) \tag{5.13b}$$

which changes the dimension of the invariant subalgebra h of $\mathbb{P}$ versus $\mathbb{A}(8;5)$. The three $\mathbb{A}$-type automorphisms in (5.12b) are related by the (trivial) inner automorphisms u and v

$$\mathbb{A}(8;5) \cong \mathbb{A} \cong \tilde{\mathbb{A}} : \quad \omega(\mathbb{A}) = u^\dagger \omega(\mathbb{A}(8;5))u, \quad \omega(\tilde{\mathbb{A}}) = v^\dagger \omega(\mathbb{A}(8;5))v \tag{5.14}$$

which act as permutations of the $\mathfrak{so}(8)$ indices and preserve $g/h = \mathfrak{so}(8)_x/(\mathfrak{so}(5)_x \oplus \mathfrak{so}(3)_{2x})$. It is known [5, 12] that conformal weights are invariant under such two-sided automorphisms, and that, indeed, all the twisted tensors of the orbifold can be taken invariant.[‡4] We may therefore consider the twisted sectors $\mathfrak{so}(8)/\mathbb{A}$ and $\mathfrak{so}(8)/\tilde{\mathbb{A}}$ as identical to the twisted sector $\mathfrak{so}(8)/\mathbb{A}(8;5)$ given above.

In what follows, we use the diagonal basis $(J_A, J^\pm_\alpha)$ discussed for $\mathbb{T}_1$ and $\mathbb{T}_2$ in Subsecs. 3.4 and 3.5.

5.4 The triality orbifold $A_{\mathfrak{so}(8)}(S_3(\mathbb{P},\mathbb{T}_1))/S_3(\mathbb{P},\mathbb{T}_1)$

To see the first S_3 triality orbifold in (5.12), we use Eqs. (3.28a), (3.32), (3.7) and (3.35) to establish the following transformation properties under $\mathbb{P}$ and $\mathbb{T}_1$

$$\omega(\mathbb{P})\left(J_A(z), J^\pm_\alpha(z)\right) = \left(J_A(z), -J^\mp_\alpha(z)\right) \tag{5.15a}$$

$$\omega(\mathbb{T}_1)\left(J_A(z), J^\pm_\alpha(z)\right) = \left(J_A(z), e^{\mp\frac{2\pi i}{3}} J^\pm_\alpha(z)\right) \tag{5.15b}$$

$$\omega(\mathbb{P}\mathbb{T}_1)\left(J_A(z), J^\pm_\alpha(z)\right) = \left(J_A(z), -e^{\mp\frac{2\pi i}{3}} J^\mp_\alpha(z)\right) \tag{5.15c}$$

$$\omega(\mathbb{T}_1\mathbb{P})\left(J_A(z), J^\pm_\alpha(z)\right) = \left(J_A(z), -e^{\pm\frac{2\pi i}{3}} J^\mp_\alpha(z)\right) \tag{5.15d}$$

$$\omega({\mathbb{T}_1}^2)\left(J_A(z), J^\pm_\alpha(z)\right) = \left(J_A(z), e^{\pm\frac{2\pi i}{3}} J^\pm_\alpha(z)\right) \tag{5.15e}$$

$$A = 1, \dots, 14, \quad \alpha = 1, \dots, 7 \tag{5.15f}$$

[‡4] See e.g. Eq. (2.28) of Ref. [5]. For our special case $U(\mathbb{A}(8;5)) = \mathbb{1}$, choose $U(\mathbb{A}) = u$ and $U(\tilde{\mathbb{A}}) = v$, with the same $E = \omega(\mathbb{A}(8;5))$.

where $\omega(CD) = \omega(C)\omega(D)$ and the index A labels the invariant $\mathfrak{g}_2$ subalgebra of $\mathbb{T}_1$. From these transformation properties, we verify that

$$\mathbb{P}^2 = {\mathbb{T}_1}^3 = (\mathbb{P}\mathbb{T}_1)^2 = 1\!\!1 \tag{5.16}$$

so $\mathbb{P}$ and $\mathbb{T}_1$ generate an S_3 whose group table is shown in Table 1 and whose conjugacy classes are

$$\begin{aligned} \sigma = 0 : &\quad 1\!\!1 \\ \sigma = 1 : &\quad \mathbb{P}, \mathbb{P}\mathbb{T}_1, \mathbb{T}_1\mathbb{P} \\ \sigma = 2 : &\quad \mathbb{T}_1, {\mathbb{T}_1}^2 . \end{aligned} \tag{5.17}$$

Then for the S_3 triality orbifold

$$\frac{A_{\mathfrak{so}(8)}(S_3(\mathbb{P}, \mathbb{T}_1))}{S_3(\mathbb{P}, \mathbb{T}_1)} \tag{5.18}$$

we may choose the twisted sectors $\mathfrak{so}(8)/\mathbb{P}$ and $\mathfrak{so}(8)/\mathbb{T}_1$ given above as the representatives for sectors $\sigma = 1$ and $\sigma = 2$ respectively.

	$\mathbb{P}$	$\mathbb{P}\mathbb{T}_1$	$\mathbb{T}_1\mathbb{P}$	$\mathbb{T}_1$	$\mathbb{T}_1^2$
$\mathbb{P}$	$1\!\!1$	$\mathbb{T}_1$	$\mathbb{T}_1^2$	$\mathbb{P}\mathbb{T}_1$	$\mathbb{T}_1\mathbb{P}$
$\mathbb{P}\mathbb{T}_1$	$\mathbb{T}_1^2$	$1\!\!1$	$\mathbb{T}_1$	$\mathbb{T}_1\mathbb{P}$	$\mathbb{P}$
$\mathbb{T}_1\mathbb{P}$	$\mathbb{T}_1$	$\mathbb{T}_1^2$	$1\!\!1$	$\mathbb{P}$	$\mathbb{P}\mathbb{T}_1$
$\mathbb{T}_1$	$\mathbb{T}_1\mathbb{P}$	$\mathbb{P}$	$\mathbb{P}\mathbb{T}_1$	$\mathbb{T}_1^2$	$1\!\!1$
$\mathbb{T}_1^2$	$\mathbb{P}\mathbb{T}_1$	$\mathbb{T}_1\mathbb{P}$	$\mathbb{P}$	$1\!\!1$	$\mathbb{T}_1$

Table 1: S_3 group table generated by $\mathbb{P}$ and $\mathbb{T}_1$.

5.5 The triality orbifold $A_{\mathfrak{so}(8)}(S_3(\mathbb{A}, \mathbb{T}_2))/S_3(\mathbb{A}, \mathbb{T}_2)$

For the second S_3 triality orbifold in (5.12), we first define the action of the outer automorphism $\mathbb{A}$ with $h = \mathfrak{so}(5)_x \oplus \mathfrak{so}(3)_{2x}$

$$J_{\dot{A}\dot{B}}(z)' = J_{\dot{A}\dot{B}}(z), \quad J_{\dot{I}\dot{J}}(z)' = J_{\dot{I}\dot{J}}(z), \quad J_{\dot{A}\dot{I}}(z)' = -J_{\dot{A}\dot{I}}(z) \tag{5.19a}$$

$$\dot{A}, \dot{B} = 2, 5, 7, \quad \dot{I}, \dot{J} = 1, 3, 4, 6, 8 \tag{5.19b}$$

which is permutation-equivalent to $\mathbb{A}(8; 5)$ in (3.22). A more convenient form of this action is

$$\omega(\mathbb{A})J_{ij}(z) = \Omega(\mathbb{A})_{ii'}\Omega(\mathbb{A})_{jj'}J_{i'j'}(z), \qquad i, j, i', j' = 1 \ldots 8 \tag{5.20a}$$

$$\Omega(\mathbb{A})_{\dot{A}\dot{B}} = \delta_{\dot{A}\dot{B}}, \quad \Omega(\mathbb{A})_{\dot{I}\dot{J}} = -\delta_{\dot{I}\dot{J}}, \quad \Omega(\mathbb{A})_{\dot{A}\dot{I}} = \Omega(\mathbb{A})_{\dot{I}\dot{A}} = 0 \tag{5.20b}$$

where $\Omega(\mathbb{A})$ is recognized as the action of charge conjugation $\mathbb{C}$ on $\mathfrak{su}(3)$ given in Ref. [7]. Then it is not difficult to check that

$$(T_A^{\text{adj}})_{kl}\Omega(\mathbb{A})_{ki}\Omega(\mathbb{A})_{lj} = \Omega(\mathbb{A})_{AB}(T_B^{\text{adj}})_{ij} \tag{5.21a}$$

$$T_i^{(3)\prime} \equiv \Omega(\mathbb{A})_{ij}T_j^{(3)} = \bar{T}_i^{(3)} \tag{5.21b}$$

$$(g_\alpha^{\pm})_{kl}\Omega(\mathbb{A})_{ki}\Omega(\mathbb{A})_{lj} = (g_\alpha^{\mp})_{ij} \tag{5.21c}$$

where $A, B, i, j = 1 \ldots 8$. Then we may use these relations and Eqs. (3.39), (3.43) to compute the actions

$$\omega(\mathbb{A})\left(J_{\dot{A}}(z), J_{\dot{I}}(z), J_\alpha^{\pm}(z)\right) = \left(J_{\dot{A}}(z), -J_{\dot{I}}(z), e^{\pm\frac{\pi i}{3}}J_\alpha^{\mp}(z)\right) \tag{5.22a}$$

$$\omega(\mathbb{T}_2)\left(J_{\dot{A}}(z), J_{\dot{I}}(z), J_\alpha^{\pm}(z)\right) = \left(J_{\dot{A}}(z), J_{\dot{I}}(z), e^{\mp\frac{2\pi i}{3}}J_\alpha^{\pm}(z)\right) \tag{5.22b}$$

$$\omega(\mathbb{A}\mathbb{T}_2)\left(J_{\dot{A}}(z), J_{\dot{I}}(z), J_\alpha^{\pm}(z)\right) = \left(J_{\dot{A}}(z), -J_{\dot{I}}(z), e^{\mp\frac{\pi i}{3}}J_\alpha^{\mp}(z)\right) \tag{5.22c}$$

$$\omega(\mathbb{T}_2\mathbb{A})\left(J_{\dot{A}}(z), J_{\dot{I}}(z), J_\alpha^{\pm}(z)\right) = \left(J_{\dot{A}}(z), -J_{\dot{I}}(z), -J_\alpha^{\mp}(z)\right) \tag{5.22d}$$

$$\omega({\mathbb{T}_2}^2)\left(J_{\dot{A}}(z), J_{\dot{I}}(z), J_\alpha^{\pm}(z)\right) = \left(J_{\dot{A}}(z), J_{\dot{I}}(z), e^{\pm\frac{2\pi i}{3}}J_\alpha^{\pm}(z)\right) \tag{5.22e}$$

$$\dot{A} = 2, 5, 7, \quad \dot{I} = 1, 3, 4, 6, 8, \quad \alpha = 1, \ldots, 10 \tag{5.22f}$$

where $\{J_A\} = \{J_{\dot{A}}, J_{\dot{I}}\}$ are the currents of the invariant $\mathfrak{su}(3)$ subalgebra of $\mathbb{T}_2$. The $\mathfrak{so}(3)_{12x} \subset \mathfrak{su}(3)_{3x}$ generated by $\{J_{\dot{A}}\}$ is not the $\mathfrak{so}(3)_{2x}$ in the invariant subalgebra of $\mathbb{A}$. From (5.22) we verify that

$$\mathbb{A}^2 = {\mathbb{T}_2}^3 = (\mathbb{A}\mathbb{T}_2)^2 = \mathbb{1} \tag{5.23}$$

and the same S_3 group table (see Table 1) and conjugacy classes (5.17) are obtained with $\mathbb{P} \to \mathbb{A}$ and $\mathbb{T}_1 \to \mathbb{T}_2$. It follows that, for the S_3 triality orbifold

$$\frac{A_{\mathfrak{so}(8)}(S_3(\mathbb{A}, \mathbb{T}_2))}{S_3(\mathbb{A}, \mathbb{T}_2)} \tag{5.24}$$

we may choose the twisted sectors $\mathfrak{so}(8)/\mathbb{A}(8;5)$ and $\mathfrak{so}(8)/\mathbb{T}_2$ given above as the representatives for sectors $\sigma = 1$ and 2 respectively.

5.6 The triality orbifold $A_{\mathfrak{so}(8)}(S_3(\tilde{\mathbb{A}}, \mathbb{T}_1))/S_3(\tilde{\mathbb{A}}, \mathbb{T}_1)$

Our last S_3 triality orbifold is the most intricate of the three.

We begin by defining the action $\omega(\tilde{\mathbb{A}})$ of the outer automorphism $\tilde{\mathbb{A}}$

$$J_{\dot{\alpha}\dot{\beta}}(z)' = J_{\dot{\alpha}\dot{\beta}}(z), \quad J_{\dot{\mu}\dot{\nu}}(z)' = J_{\dot{\mu}\dot{\nu}}(z), \quad J_{\dot{\alpha}\dot{\mu}}(z)' = -J_{\dot{\alpha}\dot{\mu}}(z) \tag{5.25a}$$

$$J_{\dot{\alpha}8}(z)' = J_{\dot{\alpha}8}, \quad J_{\dot{\mu}8}(z)' = -J_{\dot{\mu}8}(z) \tag{5.25b}$$

$$\dot{\alpha}, \dot{\beta} = 1, 2, 3, \quad \dot{\mu}, \dot{\nu} = 4, 5, 6, 7 \tag{5.25c}$$

which (like the automorphism $\mathbb{A}$) is permutation-equivalent to $\mathbb{A}(8;5)$ in (3.22). Relative to the notation of Subsec. 3.1, we have decomposed the $\mathfrak{so}(7)$ vector label μ as $\{\mu\} = \{\dot{\alpha}, \dot{\mu}\}$. A more convenient form of this action is:

$$\omega(\tilde{\mathbb{A}})J_{\mu\nu}(z) = \Omega(\tilde{\mathbb{A}})_{\mu\mu'}\Omega(\tilde{\mathbb{A}})_{\nu\nu'}J_{\mu'\nu'}(z), \qquad \mu,\nu,\mu',\nu' = 1\ldots 7 \tag{5.26a}$$

$$\omega(\tilde{\mathbb{A}})J_{\mu 8}(z) = \Omega(\tilde{\mathbb{A}})_{\mu\mu'}J_{\mu' 8}(z) \tag{5.26b}$$

$$\Omega(\tilde{\mathbb{A}})_{\dot{\alpha}\dot{\beta}} = \delta_{\dot{\alpha}\dot{\beta}}, \quad \Omega(\tilde{\mathbb{A}})_{\dot{\mu}\dot{\nu}} = -\delta_{\dot{\mu}\dot{\nu}}, \quad \Omega(\tilde{\mathbb{A}})_{\dot{\alpha}\dot{\mu}} = \Omega(\tilde{\mathbb{A}})_{\dot{\mu}\dot{\alpha}} = 0\,. \tag{5.26c}$$

Then it is not difficult to check that

$$g_{\dot{\alpha}\mu'\nu'}\Omega(\tilde{\mathbb{A}})_{\mu'\mu}\Omega(\tilde{\mathbb{A}})_{\nu'\nu} = g_{\dot{\alpha}\mu\nu}, \qquad g_{\dot{\mu}\nu'\rho'}\Omega(\tilde{\mathbb{A}})_{\nu'\nu}\Omega(\tilde{\mathbb{A}})_{\rho'\rho} = -g_{\dot{\mu}\nu\rho} \tag{5.27a}$$

$$(\rho_{\dot{A}})_{\mu'\nu'}\Omega(\tilde{\mathbb{A}})_{\mu'\mu}\Omega(\tilde{\mathbb{A}})_{\nu'\nu} = (\rho_{\dot{A}})_{\mu\nu}, \qquad (\rho_{\dot{I}})_{\mu'\nu'}\Omega(\tilde{\mathbb{A}})_{\mu'\mu}\Omega(\tilde{\mathbb{A}})_{\nu'\nu} = -(\rho_{\dot{I}})_{\mu\nu} \tag{5.27b}$$

$$\dot{A} = 1,2,5,6,11,12, \quad \dot{I} = 3,4,7,8,9,10,13,14 \tag{5.27c}$$

where $g_{\alpha\beta\gamma}$ are the octonionic structure constants in Eq. (3.27b) and $\{\rho_A\} = \{\rho_{\dot{A}}, \rho_{\dot{I}}\}$ is the trace-orthogonal $\mathfrak{g}_2$ basis given explicitly in App. C.

Finally, we use the relations (5.27) together with Eqs. (3.28a), (3.32) to compute the action of $\tilde{\mathbb{A}}$ in the $\mathbb{T}_1$ basis

$$\omega(\tilde{\mathbb{A}})\left(J_{\dot{A}}(z), J_{\dot{I}}(z), J^{\pm}_{\dot{\alpha}}(z), J^{\pm}_{\dot{\mu}}(z)\right) = \left(J_{\dot{A}}(z), -J_{\dot{I}}(z), J^{\mp}_{\dot{\alpha}}(z), -J^{\mp}_{\dot{\mu}}(z)\right) \tag{5.28a}$$

$$\omega(\mathbb{T}_1)\left(J_{\dot{A}}(z), J_{\dot{I}}(z), J^{\pm}_{\dot{\alpha}}(z), J^{\pm}_{\dot{\mu}}(z)\right) = \left(J_{\dot{A}}(z), J_{\dot{I}}(z), e^{\mp\frac{2\pi i}{3}}J^{\pm}_{\dot{\alpha}}(z), e^{\mp\frac{2\pi i}{3}}J^{\pm}_{\dot{\mu}}(z)\right) \tag{5.28b}$$

$$\omega(\tilde{\mathbb{A}}\mathbb{T}_1)\left(J_{\dot{A}}(z), J_{\dot{I}}(z), J^{\pm}_{\dot{\alpha}}(z), J^{\pm}_{\dot{\mu}}(z)\right) = \left(J_{\dot{A}}(z), -J_{\dot{I}}(z), e^{\mp\frac{2\pi i}{3}}J^{\mp}_{\dot{\alpha}}(z), -e^{\mp\frac{2\pi i}{3}}J^{\mp}_{\dot{\mu}}(z)\right) \tag{5.28c}$$

$$\omega(\mathbb{T}_1\tilde{\mathbb{A}})\left(J_{\dot{A}}(z), J_{\dot{I}}(z), J^{\pm}_{\dot{\alpha}}(z), J^{\pm}_{\dot{\mu}}(z)\right) = \left(J_{\dot{A}}(z), -J_{\dot{I}}(z), e^{\pm\frac{2\pi i}{3}}J^{\mp}_{\dot{\alpha}}(z), -e^{\pm\frac{2\pi i}{3}}J^{\mp}_{\dot{\mu}}(z)\right) \tag{5.28d}$$

$$\omega(\mathbb{T}_1{}^2)\left(J_{\dot{A}}(z), J_{\dot{I}}(z), J^{\pm}_{\dot{\alpha}}(z), J^{\pm}_{\dot{\mu}}(z)\right) = \left(J_{\dot{A}}(z), J_{\dot{I}}(z), e^{\pm\frac{2\pi i}{3}}J^{\pm}_{\dot{\alpha}}(z), e^{\pm\frac{2\pi i}{3}}J^{\pm}_{\dot{\mu}}(z)\right) \tag{5.28e}$$

$$\dot{A} = 1,2,5,6,11,12, \;\; \dot{I} = 3,4,7,8,9,10,13,14, \quad \dot{\alpha} = 1,2,3, \;\; \dot{\mu} = 4,5,6,7\,. \tag{5.28f}$$

where $\{J_A\} = \{J_{\dot{A}}, J_{\dot{I}}\}$ are the currents of the invariant $\mathfrak{g}_2$ subalgebra of $\mathbb{T}_1$ and we have also decomposed the $\mathit{7}$'s of $\mathfrak{g}_2$ as $\{J^{\pm}_{\alpha}\} = \{J^{\pm}_{\dot{\alpha}}, J^{\pm}_{\dot{\mu}}\}$. From the action (5.28), we verify that

$$\tilde{\mathbb{A}}^2 = \mathbb{T}_1{}^3 = (\tilde{\mathbb{A}}\mathbb{T}_1)^2 = 1\!\!1 \tag{5.29}$$

and the same S_3 group table (see Table 1) and conjugacy classes (5.17) are obtained with $\mathbb{P} \to \tilde{\mathbb{A}}$. It follows that, for the S_3 triality orbifold

$$\frac{A_{\mathfrak{so}(8)}(S_3(\tilde{\mathbb{A}}, \mathbb{T}_1))}{S_3(\tilde{\mathbb{A}}, \mathbb{T}_1)} \tag{5.30}$$

we may choose the twisted sectors $\mathfrak{so}(8)/\mathbb{A}(8;5)$ and $\mathfrak{so}(8)/\mathbb{T}_1$ given above as the representatives for sectors $\sigma = 1$ and 2 respectively.

In a sense, this third S_3 triality orbifold is a surprise. We have checked that the phases ${E_{\alpha_0}}' = e^{i\phi_0} E_{\alpha_0}$ of the fixed or invariant simple root operators E_{α_0} in the realization of the Dynkin automorphisms are as recorded in Fig. 1, where the solid double arrows denote the three S_3 triality orbifolds we have now constructed. The diagram shows clearly that the relative simplicity seen above for the vertical pairings is correlated with paired trivial or paired non-trivial phases of the fixed simple root operators. For the third orbifold, however, we have mixed a trivial phase for the $\mathbb{Z}_3$ with a non-trivial phase for the $\mathbb{Z}_2$. The phases of the $\mathbb{Z}_2$ automorphisms are further discussed in App. E.

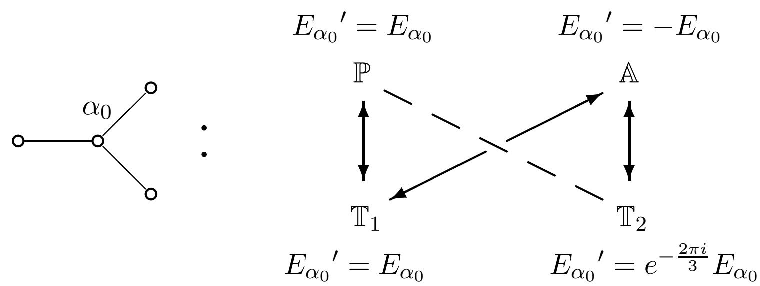

Fig. 1: Phases of ${E_{\alpha_0}}'$ for the S_3 triality orbifolds.

The diagram in Fig. 1 also suggests the existence of a fourth S_3 triality orbifold corresponding to the dashed line, but we have proven (see App. E) that no such fourth S_3 orbifold can be constructed.

Although all our orbifolds are consistent on the sphere, they should also be checked against modular invariance on the torus – but this is beyond the scope of the present paper.

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A Outer automorphisms and invariant subalgebras

	g/h
i	$A_{2r}/B_r = \mathfrak{su}(2r+1)/\mathfrak{so}(2r+1)$
ii	$A_{2r-1}/D_r = \mathfrak{su}(2r)/\mathfrak{so}(2r)$
iii	$A_{2r-1}/C_r = \mathfrak{su}(2r)/C_r{=}\mathfrak{su}(2r)/\mathfrak{sp}(r)$
iv	$D_{r+1}/B_r = \mathfrak{so}(2r+2)/\mathfrak{so}(2r+1)$
v	$D_{r+1}/[B_n \oplus B_{r-n}] = \mathfrak{so}(2r+2)/[\mathfrak{so}(2n+1) \oplus \mathfrak{so}(2(r-n)+1)]$
vi	E_6/F_4
vii	E_6/C_4
viii	$D_4/G_2 = \mathfrak{so}(8)/\mathfrak{g}_2$
ix	$D_4/A_2 = \mathfrak{so}(8)/\mathfrak{su}(3)$

In the form g/h, this table gives the inequivalent realizations (homogeneous gradations) of the Dynkin automorphisms of simple g in terms of their invariant subalgebras h. To our knowledge this information appeared first in Table 5 of Ref. [27]. The first seven entries of the table are of type $\mathbb{Z}_2$ ($\omega^2 = 1$), where g/h is a symmetric space, while the last two entries are of type $\mathbb{Z}_3$ ($\omega^3 = 1$). Different entries for the same g are inner-automorphically equivalent (see e.g. Ref. [13]) because each entry represents the same Dynkin automorphism of g. Nevertheless, the twisted sector corresponding to each entry on each g is physically distinct.

The cases i) and ii) complete to $\mathfrak{su}(n)/\mathfrak{so}(n)$, identified in Ref.[13] as the charge conjugation automorphism $\mathbb{C}$ on $\mathfrak{su}(n)$, and the corresponding charge conjugation orbifold on $\mathfrak{su}(n)$ is also discussed in that reference. The present paper discusses the twisted sectors and orbifolds corresponding to all the outer automorphisms on $\mathfrak{so}(2n)$, including $\mathbb{P}$ (entry iv), $\{\mathbb{A}(2n; r), r = \text{odd}\}$ (entry v), $\mathbb{T}_1$ (entry viii) and $\mathbb{T}_2$ (entry ix).

When realizations differ on the same g, the difference is in the phases of the fixed simple root operators under the Dynkin automorphism (see e.g. Ref. [13]). As an example consider $\mathfrak{so}(2n \geq 6)$ in the Cartan-Weyl basis, with simple roots and Cartan generators:

$$\alpha_{(i)} = e_i - e_{i+1}, \qquad i = 1, \ldots, n-1, \qquad \alpha_{(n)} = e_{n-1} + e_n \tag{A.1a}$$

$$h_i \equiv e_i \cdot H, \qquad i = 1 \ldots n \; . \tag{A.1b}$$

Then we have checked in particular that explicit realizations of the outer automorphisms $\mathbb{P}$ and $\mathbb{A} \equiv \mathbb{A}(2n; 2n-3)$ may be completed by starting from the actions on the simple root operators

$$\mathbb{P}\left(\frac{\mathfrak{so}(2n)}{\mathfrak{so}(2n-1)}\right) : \qquad \begin{array}{cc} E_{\alpha_{(i)}}{}' = E_{\alpha_{(i)}}, & i = 1, \ldots, n-2 \\ E_{\alpha_{(n-1)}}{}' = E_{\alpha_{(n)}}, & E_{\alpha_{(n)}}{}' = E_{\alpha_{(n-1)}} \end{array} \tag{A.2a}$$

$$\mathbb{A}\left(\frac{\mathfrak{so}(2n)}{\mathfrak{so}(2n-3)\oplus\mathfrak{so}(3)}\right)\ :\qquad \begin{array}{c} E_{\alpha_{(i)}}{}' = E_{\alpha_{(i)}}, \qquad i=1,\ldots,n-3, \qquad E_{\alpha_{(n-2)}}{}' = -E_{\alpha_{(n-2)}} \\ E_{\alpha_{(n-1)}}{}' = E_{\alpha_{(n)}}, \qquad E_{\alpha_{(n)}}{}' = E_{\alpha_{(n-1)}} \end{array} \tag{A.2b}$$

and $h_i{}' = h_i$, $i = 1,\ldots n-1$, $h_n{}' = -h_n$ for both cases. This computation verifies that the series in entry v) of the table is outer-automorphic down to and including $B_1 \equiv \mathfrak{so}(3)$. We also give the generators of the invariant $B_1 = \mathfrak{so}(3) \subset (\mathfrak{so}(2n-3) \oplus \mathfrak{so}(3))$ under the action of $\mathbb{A}$

$$\{E_{\pm\alpha_{(n-1)}} + E_{\pm\alpha_{(n)}}, h_{n-1}\} \tag{A.3}$$

because this case is of special interest in the text (See Subsecs. 3.3, 5.5 and 5.6). For the outer automorphisms on $\mathfrak{so}(8)$, the phases for the fixed simple root operators E_{α_0} are shown in Fig. 1.

B Spinor reps of $\mathfrak{spin}(2n)$

We consider as examples some standard [28] constructions of Dirac and Weyl spinor reps of $\mathfrak{spin}(2n)$.

In the first example, the Dirac gamma matrices $\{\gamma_i\}$ are constructed from n complex fermions as follows

$$\{b_i, b_j^\dagger\} = \delta_{ij}, \qquad \{b_i, b_j\} = \{b_i^\dagger, b_j^\dagger\} = 0, \qquad i,j = 1\ldots n \tag{B.1a}$$

$$\gamma_i \equiv \begin{cases} b_i + b_i^\dagger & i = 1\ldots n \\ \frac{1}{i}(b_{i-n} - b_{i-n}^\dagger) & i = n+1,\ldots,2n \end{cases}, \qquad \{\gamma_i, \gamma_j\} = 2\delta_{ij}, \qquad i,j = 1\ldots 2n \tag{B.1b}$$

$$\gamma_i = \gamma_i^\dagger = \begin{pmatrix} 0 & \Gamma_i \\ \Gamma_i^\dagger & 0 \end{pmatrix}, \qquad \Gamma_i\Gamma_j^\dagger + \Gamma_j\Gamma_i^\dagger = \Gamma_i^\dagger\Gamma_j + \Gamma_j^\dagger\Gamma_i = 2\delta_{ij} \tag{B.1c}$$

$$\gamma_i^* = \gamma_i^t = \begin{cases} \gamma_i & i = 1\ldots n \\ -\gamma_i & i = n+1,\ldots,2n \end{cases} \tag{B.1d}$$

where t is matrix transpose. This gives the Dirac spinor rep $\{D_{ij}\}$ of $\mathfrak{spin}(2n)$ and its decomposition into the Weyl reps S_{ij} and C_{ij}:

$$D_{ij} = D_{ij}^\dagger = \frac{i}{4}[\gamma_i, \gamma_j] = \begin{pmatrix} S_{ij} & 0 \\ 0 & C_{ij} \end{pmatrix} \tag{B.2a}$$

$$S_{ij} = S_{ij}^\dagger = \frac{i}{4}(\Gamma_i\Gamma_j^\dagger - \Gamma_j\Gamma_i^\dagger), \qquad C_{ij} = C_{ij}^\dagger = \frac{i}{4}(\Gamma_i^\dagger\Gamma_j - \Gamma_j^\dagger\Gamma_i)\ . \tag{B.2b}$$

Each of these reps satisfy the algebra of $\mathfrak{so}(2n) \cong \mathfrak{spin}(2n)$

$$[T_{ij}, T_{kl}] = i(\delta_{jk}T_{il} - \delta_{ik}T_{jl} - \delta_{jl}T_{ik} + \delta_{il}T_{jk}), \qquad T = D, S \text{ or } C \tag{B.3}$$

as expected.

We know from the text that the automorphic transforms T' of T

$$T' = \omega T, \qquad T = D, S \text{ or } C, \qquad \omega = \omega(\mathbb{A}(2n;r)), \qquad r = n, \dots, 2n-1 \tag{B.4}$$

also satisfy Eq. (B.3). For these reps we find that $\bar{T} = T'$ when prime is charge conjugation $\mathbb{C} = \mathbb{A}(2n;n)$

$$\begin{aligned} (\bar{T})_{ij} = -(T^{\mathrm{t}})_{ij} &= \begin{cases} T_{ij} & i,j = 1\dots n \text{ or } i,j = n+1,\dots,2n \\ -T_{ij} & i = 1\dots n,\ j = n+1,\dots,2n \end{cases} && \text{(B.5a)} \\ &= T_{ij}{}', \qquad T = D, S \text{ or } C && \text{(B.5b)} \end{aligned}$$

where ω for $\mathbb{C}$ is given in (3.13a). More generally, charge conjugation gives $T' \cong \bar{T}$ for all irreps of any Lie g. Together, (3.18) and the result (B.5) tell us that

$$\mathfrak{spin}(4r+2), \qquad \mathbb{C} \simeq \mathbb{P}: \qquad S_{ij}{}' = \bar{S}_{ij} \cong C_{ij}, \qquad C_{ij}{}' = \bar{C}_{ij} \cong S_{ij} \tag{B.6a}$$

$$\mathfrak{spin}(4r), \qquad \mathbb{C} \not\simeq \mathbb{P}: \qquad S_{ij}{}' = \bar{S}_{ij} \cong S_{ij}, \qquad C_{ij}{}' = \bar{C}_{ij} \cong C_{ij} \tag{B.6b}$$

where prime is charge conjugation. In the language of the text, the Weyl reps are real under $\mathbb{C}$ for $\mathfrak{spin}(4r)$ and complex under $\mathbb{C}$ for $\mathfrak{spin}(4r+2)$.

Other representations can be more natural for other automorphisms. Consider for example the well-known [28] Dirac spinor rep of $\mathfrak{spin}(8)$:

$$\gamma_i = \gamma_i^\dagger = \gamma_i^{\mathrm{t}} = \begin{pmatrix} 0 & \Gamma_i \\ \Gamma_i^\dagger & 0 \end{pmatrix}, \qquad \Gamma_i^* = \Gamma_i, \qquad \Gamma_i^{\mathrm{t}} = \begin{cases} -\Gamma_i & i = 1\dots 7 \\ \Gamma_8 & i = 8 \end{cases} \tag{B.7a}$$

$$\epsilon = i\tau_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{B.7b}$$

$$\Gamma_1 = \epsilon \otimes \epsilon \otimes \epsilon, \qquad \Gamma_2 = 1 \otimes \tau_1 \otimes \epsilon, \qquad \Gamma_3 = 1 \otimes \tau_3 \otimes \epsilon, \qquad \Gamma_4 = \tau_1 \otimes \epsilon \otimes 1 \tag{B.7c}$$

$$\Gamma_5 = \tau_3 \otimes \epsilon \otimes 1, \qquad \Gamma_6 = -\epsilon \otimes 1 \otimes \tau_1, \qquad \Gamma_7 = \epsilon \otimes 1 \otimes \tau_3, \qquad \Gamma_8 = 1 \otimes 1 \otimes 1 \tag{B.7d}$$

where the Weyl spinor reps S and C are again computed from Eq. (B.2). In this case we verify that $S' = C$ and $C' = S$

$$\begin{aligned} C_{ij} &= \frac{i}{4}(\Gamma_i^\dagger \Gamma_j - \Gamma_j^\dagger \Gamma_i) = \begin{cases} S_{ij} & i,j = 1\dots 7 \\ -S_{i8} & j = 8 \end{cases} && \text{(B.8a)} \\ &= S_{ij}{}' && \text{(B.8b)} \\ S_{ij} &= C_{ij}{}' && \text{(B.8c)} \end{aligned}$$

where prime is the parity automorphism $\mathbb{P}$ in (3.7).

C More about the embedding $\mathfrak{so}(8)_x \supset \mathfrak{so}(7)_x \supset (\mathfrak{g}_2)_x$

A trace-orthogonal set of solutions $\{\rho_A, A = 1 \dots 14\}$ of Eq. (3.28b) is

$$\rho_1 = \frac{1}{\sqrt{2}}(e_{23} - e_{67}), \qquad \rho_2 = \frac{1}{\sqrt{6}}(e_{23} + e_{67} - 2e_{45}), \qquad \rho_3 = \frac{1}{\sqrt{2}}(e_{37} - e_{14}) \tag{C.1a}$$

$$\rho_4 = \frac{1}{\sqrt{6}}(e_{37} + e_{14} + 2e_{26}), \qquad \rho_5 = \frac{1}{\sqrt{2}}(e_{47} - e_{56}), \qquad \rho_6 = \frac{1}{\sqrt{6}}(e_{47} + e_{56} + 2e_{13}), \tag{C.1b}$$

$$\rho_7 = \frac{1}{\sqrt{2}}(e_{16} - e_{24}), \qquad \rho_8 = \frac{1}{\sqrt{6}}(e_{16} + e_{24} + 2e_{35}), \qquad \rho_9 = \frac{1}{\sqrt{2}}(e_{25} - e_{34}) \tag{C.1c}$$

$$\rho_{10} = \frac{1}{\sqrt{6}}(e_{25} + e_{34} + 2e_{17}), \qquad \rho_{11} = \frac{1}{\sqrt{2}}(e_{12} - e_{46}), \qquad \rho_{12} = \frac{1}{\sqrt{6}}(e_{12} + e_{46} + 2e_{57}) \tag{C.1d}$$

$$\rho_{13} = \frac{1}{\sqrt{2}}(e_{36} - e_{27}), \qquad \rho_{14} = \frac{1}{\sqrt{6}}(e_{36} + e_{27} - 2e_{15}) \tag{C.1e}$$

where the antisymmetric tensors $\{e_{ij}\}$ are defined in (3.1b). Then using $J_A = (\rho_A) \cdot J$ and the second part of (3.29), one obtains the $\mathfrak{g}_2$ OPE (3.33a) and the following OPEs

$$J_A(z) J_\alpha(w) = \frac{i f_{A\alpha\beta} J_\beta(w)}{z - w} + \mathcal{O}(z-w)^0, \qquad f_{A\alpha\beta} = -2(\rho_A)_{\alpha\beta} \tag{C.2a}$$

$$J_A(z) J_{\alpha 8}(w) = \frac{i f_{A\alpha\beta} J_{\beta 8}(w)}{z - w} + \mathcal{O}(z-w)^0 \tag{C.2b}$$

$$J_{\alpha 8}(z) J_{\beta 8}(w) = \frac{k \delta_{\alpha\beta}}{(z-w)^2} - \frac{i J_{\alpha\beta}(w)}{z - w} + \mathcal{O}(z-w)^0 \tag{C.2c}$$

$$J_\alpha(z) J_\beta(w) = \frac{12 k \delta_{\alpha\beta}}{(z-w)^2} + \frac{2i(g_{\alpha\mu\gamma} g_{\beta\gamma\nu} - g_{\beta\mu\gamma} g_{\alpha\gamma\nu}) J_{\mu\nu}(w)}{z - w} + \mathcal{O}(z-w)^0 \tag{C.2d}$$

$$J_\alpha(z) J_{\beta 8}(w) = J_{\alpha 8}(z) J_\beta(w) = \frac{2i g_{\alpha\beta\gamma} J_{\gamma 8}(w)}{z - w} + \mathcal{O}(z-w)^0 \tag{C.2e}$$

by direct computation. The rest of the OPEs in (3.33) then follow from (3.32) and (C.2).

In computing (3.33c) we also find an operator term proportional to $J_{\mu\nu}$, but the coefficient is

$$6(e_{\alpha\beta})_{\mu\nu} - 2g_{\alpha\beta\gamma} g_{\mu\nu\gamma} - g_{\alpha\mu\gamma} g_{\beta\nu\gamma} + g_{\beta\mu\gamma} g_{\alpha\nu\gamma} = 0 \ . \tag{C.3}$$

A consequence of this sum rule is the identity

$$b_\alpha d_\beta d_\rho d_\sigma g_{\alpha\beta\gamma} g_{\rho\sigma\gamma} = b^2 d^2 - (b \cdot d)^2 \tag{C.4}$$

for arbitrary 7-vectors b and d. The identity (C.4) allows us to verify the familiar lemma

$$|AB| = |A||B| = \sqrt{(a^2 + b^2)(c^2 + d^2)} \tag{C.5}$$

for arbitrary octonions A and B with modulus $|a + b \cdot i| = \sqrt{a^2 + b^2}$.

Finally, we consider (3.33d). In this case, the terms proportional to $J_{\alpha 8}$ cancel and the term proportional to $J_{\mu\nu}$ has the coefficient

$$-\frac{1}{2}(e_{\alpha\beta})_{\mu\nu} + \frac{1}{12}(g_{\alpha\mu\gamma} g_{\beta\gamma\nu} - g_{\beta\mu\gamma} g_{\alpha\gamma\nu}) \tag{C.6a}$$

$$= -(e_{\alpha\beta})_{\mu\nu} + \frac{1}{6} g_{\alpha\beta\gamma} g_{\mu\nu\gamma} \tag{C.6b}$$

$$= -2(\rho_A)_{\alpha\beta}(\rho_A)_{\mu\nu} \tag{C.6c}$$

where we have used the sum rule (C.3) to obtain (C.6b). The last identity in (C.6c) shows that the operator term is in $\mathfrak{g}_2$ and verifies that the totally antisymmetric structure constants satisfy (3.34) as they should. Octonionic identities of the type shown in (C.3) and (C.6c) are discussed from a different point of view in Ref. [20]

D More about the embedding $\mathfrak{so}(8)_x \supset \mathfrak{su}(3)_{3x}$

We list here some identities which we need to obtain the diagonal basis of the triality automorphism $\mathbb{T}_2$ in Subsec. 3.5. The basic properties of the $\mathfrak{z}$ of $\mathfrak{su}(3)$ are

$$T_A^{(3)} = \frac{\sqrt{\psi^2_{\mathfrak{su}(3)}}}{2} \lambda_A, \qquad \mathrm{Tr}\left(T_A^{(3)}\right) = 0, \qquad \mathrm{Tr}\left(T_A^{(3)} T_B^{(3)}\right) = \frac{\psi^2_{\mathfrak{su}(3)}}{2} \delta_{AB}, \qquad A, B = 1 \ldots 8 \tag{D.1a}$$

$$\sum_{A=1}^{8} \left(T_A^{(3)}\right)_{IJ} \left(T_A^{(3)}\right)_{KL} = \frac{\psi^2_{\mathfrak{su}(3)}}{2} (\delta_{IL}\delta_{JK} - \frac{1}{3}\delta_{IJ}\delta_{KL}), \qquad I, J, K, L = 1, 2, 3 \tag{D.1b}$$

where $\{\lambda_A\}$ are the Gell-Mann matrices and $\psi^2_{\mathfrak{su}(3)}$ is the root length squared of $\mathfrak{su}(3)$.

Using the definitions in the text, we find the identities

$$\left(T_A^{\mathrm{adj}}\right)^t = -T_A^{\mathrm{adj}}, \qquad \left(g_\alpha^\pm\right)^t = -g_\alpha^\pm, \quad \alpha = 1, \ldots, 10 \tag{D.2a}$$

$$\mathrm{Tr}\left(T_A^{(10)}\right) = \mathrm{Tr}\left(T_A^{\mathrm{adj}} g_\alpha^\pm\right) = \mathrm{Tr}\left(g_\alpha^\pm g_\beta^\pm\right) = 0, \qquad \mathrm{Tr}\left(g_\alpha^\pm g_\beta^\mp\right) = -18\psi^4_{\mathfrak{su}(3)} \mathbb{1}_{\alpha\beta} \tag{D.2b}$$

where t is matrix transpose. The matrix $\mathbb{1}$ in (3.47b) is the true identity matrix in the 10×10 space because

$$\mathbb{1}_{IJK;LMN} B_{LMN} = B_{IJK}, \qquad \alpha = \{IJK\} = 1, \dots, 10 \tag{D.3}$$

for any symmetric rank 3 tensor B.

We also find the commutation relations

$$[T_A^{\mathrm{adj}}, g_\alpha^+] \;=\; -\left(T_A^{(\mathbf{10})}\right)_{\alpha\beta} g_\beta^+, \qquad [T_A^{\mathrm{adj}}, g_\alpha^-] = -\left(\overline{T}_A^{(\mathbf{10})}\right)_{\alpha\beta} g_\beta^- \tag{D.4a}$$

$$[g_\alpha^+, g_\beta^-] \;=\; 6\psi^2_{\mathfrak{su}(3)} \left(T_A^{(\mathbf{10})}\right)_{\alpha\beta} T_A^{\mathrm{adj}}, \qquad [g_\alpha^\pm, g_\beta^\pm] = -3\psi^2_{\mathfrak{su}(3)} \widetilde{g}_{\alpha\beta\gamma} g_\gamma^\mp \tag{D.4b}$$

and the tensor $\widetilde{g}_{\alpha\beta\gamma}$ (see (3.47a)) is fully antisymmetric. Also useful are the sum rules

$$\sum_\alpha \left\{ \left(g_\alpha^+\right)_{ij} \left(g_\alpha^-\right)_{kl} + \left(g_\alpha^-\right)_{ij} \left(g_\alpha^+\right)_{kl} \right\} = 18\psi^4_{\mathfrak{su}(3)} (e_{ij})_{kl} + 6\psi^2_{\mathfrak{su}(3)} \sum_A \left(T_A^{\mathrm{adj}}\right)_{ij} \left(T_A^{\mathrm{adj}}\right)_{kl} \tag{D.5a}$$

$$\widetilde{g}_{\alpha\gamma\delta} \widetilde{g}_{\gamma\delta\beta} = \sum_A (T_A^{(\mathbf{10})} T_A^{(\mathbf{10})})_{\alpha\beta} = \sum_A (\bar{T}_A^{(\mathbf{10})} \bar{T}_A^{(\mathbf{10})})_{\alpha\beta} \;. \tag{D.5b}$$

In the text, the identities of this appendix are used with $\psi^2_{\mathfrak{su}(3)} = \frac{2}{3}$.

E Nonexistence of a fourth S_3 triality orbifold on $\mathfrak{so}(8)$

In Sec. 5, we constructed three S_3 triality orbifolds on $\mathfrak{so}(8)$ and in Subsec. 5.6 we mentioned the possibility of a fourth S_3 triality orbifold on $\mathfrak{so}(8)$ based on any $\tilde{\mathbb{P}} \cong \mathbb{P}$ and the triality automorphism $\mathbb{T}_2$. We now present a proof that no such S_3 can be constructed.

We begin by noting some properties of the $\mathbb{Z}_2$ outer automorphism $\tilde{\mathbb{P}}$: Since it is a $\mathbb{Z}_2$ automorphism, all of the eigenvalues of $\omega(\tilde{\mathbb{P}})$ are ± 1. Furthermore, we know that any outer automorphism $\tilde{\mathbb{P}} \cong \mathbb{P}$ on $\mathfrak{so}(8)$ will leave an $\mathfrak{so}(7)$ subalgebra invariant. Therefore, $\omega(\tilde{\mathbb{P}})$ has eigenvalue $+1$ with multiplicity 21, and eigenvalue -1 with multiplicity 7. As we shall see, it is this fact which makes it impossible to generate an S_3 from $\tilde{\mathbb{P}}$ and $\mathbb{T}_2$.

To understand this, let us assume the opposite, namely that it is possible to generate an S_3 from $\tilde{\mathbb{P}}$ and $\mathbb{T}_2$. Then we have the following necessary and sufficient identities:

$${\mathbb{T}_2}^3 = \tilde{\mathbb{P}}^2 = (\tilde{\mathbb{P}}\mathbb{T}_2)^2 = \mathbb{1} \tag{E.1a}$$

$$\Rightarrow \omega(\tilde{\mathbb{P}})^2 = \mathbb{1}, \quad \omega(\tilde{\mathbb{P}})\omega(\mathbb{T}_2) = \omega(\mathbb{T}_2)^2 \omega(\tilde{\mathbb{P}}) \,. \tag{E.1b}$$

In the $\mathbb{T}_2$ basis developed in Subsec. 3.5, we know the explicit, diagonal form of $\omega(\mathbb{T}_2)$

$$\omega(\mathbb{T}_2) = \begin{pmatrix} \mathbb{1}_8 & 0 & 0 \\ 0 & \eta \mathbb{1}_{10} & 0 \\ 0 & 0 & \eta^2 \mathbb{1}_{10} \end{pmatrix}, \quad \eta = e^{-\frac{2\pi i}{3}} \tag{E.2}$$

where the diagonal blocks correspond to the action on the currents $(J_A(z), J_\alpha^+(z), J_\alpha^-(z))$ respectively. Using Eq. (E.2), we find that the general solution $\omega(\tilde{\mathbb{P}})$ of the relations (E.1b) is

$$\omega(\tilde{\mathbb{P}}) = \begin{pmatrix} A_8 & 0 & 0 \\ 0 & 0 & B_{10} \\ 0 & C_{10} & 0 \end{pmatrix}, \quad A_8^2 = \mathbb{1}_8, \quad B_{10}C_{10} = \mathbb{1}_{10} \tag{E.3}$$

and so the 20×20 submatrix $\tilde{B}$ satisfies

$$\tilde{B} \equiv \begin{pmatrix} 0 & B_{10} \\ C_{10} & 0 \end{pmatrix}, \quad \tilde{B}^2 = \mathbb{1}, \quad \mathrm{Tr}(\tilde{B}) = 0\,. \tag{E.4}$$

It follows that $\tilde{B}$ has 10 eigenvalues $+1$ and 10 eigenvalues -1, and hence $\omega(\tilde{\mathbb{P}})$ (if it exists) has *at least 10 eigenvalues* -1. We have now reached a contradiction because, as we explained above, $\omega(\tilde{\mathbb{P}})$ must have exactly 7 eigenvalues -1. It follows that there is no fourth S_3 triality orbifold on $\mathfrak{so}(8)$.

We note in passing that the other $\mathbb{Z}_2$-type automorphism, namely $\omega(\mathbb{A})$, passes the S_3 test above as it must: Because $\omega(\mathbb{A})$ leaves $\mathfrak{so}(3) \oplus \mathfrak{so}(5)$ invariant, it has 13 eigenvalues $+1$ and 15 eigenvalues -1.

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